Coactions of Hopf-C^*-algebras and equivariant E-theory

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Abstract

We define and study an equivariant E-theory with respect to coactions of Hopf C^*-algebras; we prove the Baaj-Skandalis duality in this setting. We show that the corresponding equivariant KK-theory of Baaj and Skandalis enjoys an universal property. In the appendix, we look at the different ways of expressing equivariant stability for a functor, and prove an equivariant Brown-Green-Rieffel stabilization result.

Contents

1 Introduction

Assume that A is a C^*-algebra acted upon by a locally compact abelian group G; the cross product $A \rtimes G$ is endowed with an action of the dual group of G, $\hat{G}$, and the double cross product $A \rtimes G \rtimes \hat{G}$ is isomorphic to $A \otimes \mathcal{K}(L^2(G))$. This isomorphism,

the Takesaki-Takay duality, was generalized to noncommutative groups, and roughly states that the cross product $A \rtimes_r G$ is endowed with a coaction of G, and a similar duality holds by taking the cross product by the coaction of G. When G is abelian, coactions of G correspond to actions of the dual group $\hat{G}$, so one recovers the same isomorphism.

Hopf C^*-algebras provide a common framework for actions and coactions of groups and their cross products, and for duality results. It also allows us to prove duality beyond the case of groups, in the realm of quantum groups. In the general setting, Hopf C^*-algebras were considered in K-theory by Baaj and Skandalis who defined and studied an equivariant KK-theory with respect to coactions of Hopf C^*-algebras. This theory is a generalization of Kasparov's equivariant KK-theory for groups: when the Hopf C^*-algebra in question is $C_0(G)$, the Hopf algebra associated to a group G, there is an identification between $KK^{C_0(G)}$ and KK^G.

One of the main properties of this theory is the Baaj-Skandalis isomorphism which identifies the group $KK^G(A, B)$ with the group $KK^{C_r^*(G)}(A \rtimes_r G, B \rtimes_r G)$; there is also a dual isomorphism.

This Baaj-Skandalis isomorphism is a valuable tool in the study of the K-theory of cross product algebras, and recently J. Cuntz used it in connection to the Baum-Connes conjecture which predicts that the K-theory of the cross product $A \rtimes_r G$ is isomorphic, by means of an index map, to a group of generalized elliptic operators with coefficients in A. In [8] Cuntz reinterpreted the above conjecture for discrete groups as follows: using the Baaj-Skandalis isomorphism he defined a finitely supported K-theory, $K_*^{fin}(A \rtimes_r G)$, for the cross product algebras $A \rtimes_r G$, and reformulated the Baum-Connes assembly map as an application

$$\mu : K_*^{fin}(A \rtimes_r G) \to K_*(A \rtimes_r G)$$

between finitely supported and ordinary K-theory of the cross product algebras $A \rtimes_r G$.

In this article we define the equivariant E-theory for coactions of Hopf C^*-algebras. Unless stated otherwise, we use the minimal tensor product of C^*-algebras and the reduced cross product.

E-theory, which is due to Connes and Higson, associates to a pair of C^*-algebras (A, B), an abelian group $E(A, B)$. It is a contravariant functor in the first variable and a covariant one in the second, and there is a product $E(A, B) \times E(B, C) \to E(A, C)$ which allows us to see it as a category. All these properties are also shared by the Kasparov's KK-theory of which E-theory is a variant : there is a natural transformation from KK to E-theory, and various results explain the relations and the differences between these theories.

Equivariant E-theory with respect to group actions was defined by Guentner, Higson and Trout ([13]) and was used by Higson and Kasparov in their proof of the Baum-Connes conjecture for amenable groups ([15]).

An important feature of E-theory is that, contrary to KK-theory, it does not behave well with respect to minimal tensor product of C^*-algebras or reduced cross products by groups, mainly because these operations are not exact, as explain in Section 6.3 of [12], based on the argument from [23]. But E-theory is based on the following simple to state notion, introduced by Connes and Higson in [9]:

Definition 1.1 Let $T = [1, +\infty[$. An asymptotic morphism between two C^*-algebras A and B is a family of maps $(\varphi_t)_{t \in T} : A \to B$ such that

(i) $\varphi_t(a + \lambda b) - \varphi_t(a) - \lambda \varphi_t(b) \to 0$

(ii) $\varphi_t(ab) - \varphi_t(a) \varphi_t(b) \to 0$

(iii) $\varphi_t(a)^* - \varphi_t(a^*) \to 0$

as $t \to +\infty$ and such that for every $a \in A$, $t \mapsto \varphi_t(a)$ is continuous.

Two asymptotic morphisms $(\varphi_t)_{t \in T}$ and $(\varphi'_t)_{t \in T}$ are (asymptotically) equivalent if for every $a \in A$, $(\varphi_t - \varphi'_t)(a) \to 0$ as $t \to +\infty$. Two asymptotic morphisms $\varphi^0_t, \varphi^1_t : A \to B$ are *homotopic* if they are obtained from an asymptotic morphism $\Phi : A \to C([0, 1], B)$ by evaluation in 0 and in 1, respectively. Homotopy is an equivalence relation, denote by $\{A, B\}$ the homotopy classes of asymptotic morphisms between A and B.

A different picture of this is obtained as follows.

Definition 1.2 Let A be a C^*-algebra, the asymptotic algebra of A, $\mathfrak{A}A = C_b(T, A)/C_0(T, A)$ is the quotient of the C^*-algebra of bounded continuous functions on T with values in A by the ideal of functions vanishing at infinity. Denote by $\alpha_A : A \to \mathfrak{A}A$ the map that associates to an element $a \in A$ the class of the constant map $t \mapsto a$.

There is a one-to-one correspondence between equivalence classes of asymptotic morphisms and *-homomorphisms $\varphi : A \to \mathfrak{A}B$. We shall follow [13] and use this point of view, rather then the initial approach from [9]. Let $\varphi : A \to B$ be a $*$-homomorphism between the C^*-algebras A and B; composition with φ induces a map $\mathfrak{A}\varphi : \mathfrak{A}A \to \mathfrak{A}B$, so one can regard $\mathfrak{A}$ as a functor on the category of C^*-algebras. We use the notation $\mathfrak{A}^n$ for the n-fold composition of $\mathfrak{A}$ with itself for $n \geq 1$ and the identity functor for $n = 0$.

A summary of this article is as follows.

We start the first section by recalling some facts and by fixing some notations about multiplier algebras, Hopf C^*-algebras S, and coactions of them on C^*-algebras. Details can be found in [1] and in [11]. We define the notion of an S-asymptotic morphism and $\mathfrak{A}_S A$, the subalgebra of S-continuous elements of $\mathfrak{A}A$. We show that the asymptotic morphism associated by the Connes-Higson construction to an S-equivariant extension is equivariant in the sense that we had defined.

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In the second part we construct a category which combines homotopy classes of equivariant asymptotic morphism with exterior equivalence of coactions. We keep track of this using cocycles for the coactions involved.

Then we define the equivariant E-theory with respect to coactions of Hopf C^*-algebras and prove its main properties:existence of a product, six-exact sequences, tensor product. We also show that, when the Hopf algebra is associated to the action of a locally compact group, i.e., when $S = C_0(G)$, we can identify our $E^{C_0(G)}$ with the equivariant E-theory for groups, E^G, previously defined by Gueunter, Higson and Trout in [13]. We obtain in this way an alternative description of their theory. We prove the Baaj-Skandalis isomorphism in E-theory.

In the fourth section we study the equivariant KK-theory from the point of view of its universal property and derive the existence of a natural transformation from KK^S to E^S.

In the appendix we prove an equivariant version of the a theorem of Brown, Green, Rieffel, which states that two Morita equivalent separable C^*-algebras are stable isomorphic. As a consequence one obtains that different ways of expressing stability of a functor are equivalent.

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2 Equivariant asymptotic morphisms

Let A be a C^*-algebra, denote by $M(A)$ the associated multiplier algebra, the biggest C^*-algebra which contains A as an essential ideal. If J is an indeal in A, it follows that there is a restriction map from $M(J)$ to $M(A)$; this map is neither injective nor surjective, in general.

$M(A)$ is the completion of A with respect to the strict topology, the topology generated by the seminorms $p_c(a) = \|ac\| + \|ca\|$, indexed by elements c in A. Recall that a morphism $\varphi : A \to B$ is *nondegenerate* if there is an approximate unit $(e_i)_i$ for A such that $\varphi(e_i)$ strictly converges to $1_{M(B)}$. Such a morphism φ extends to a map between the multiplier algebras, a map that we denote again by $\varphi : M(A) \to M(B)$.

Let A and S be C^*-algebras, denote by

$$\tilde{M}(A \otimes S) = \{m \in M(A \otimes S) : m(1 \otimes S) + (1 \otimes S)m \subset A \otimes S\},$$

the S-multipliers of $A \otimes S$; this is a closed *-subalgebra of $M(A \otimes S)$.

The S-strict topology of $\tilde{M}(A \otimes S)$ is the locally convex topology generated by the seminorms $p_s(m) = \|m(1 \otimes s)\| + \|(1 \otimes s)m\|$ indexed by $s \in S$.

The relevance of this topology for $\tilde{M}(A \otimes S)$ was observed in [11], Proposition A.4, where it is proved that:

Lemma 2.1 $\tilde{M}(A \otimes S)$ *is the S-strict completion of $A \otimes S$.*

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Examples 1. If S is unital, then $\tilde{M}(A \otimes S) = A \otimes S$.

2. Assume that A is unital, then $\tilde{M}(A \otimes S) = M(A \otimes S)$.

3. When $S = C_0(X)$ the continuous functions vanishing at infinity on a locally compact Hausdorff space X, then $\tilde{M}(A \otimes C_0(X)) = C_b(X, A)$, the algebra of bounded continuous functions with values in A. Note that the multiplier algebra $M(A \otimes C_0(X))$ is $C_b(X, M^s(A))$ the algebra of bounded strictly continuous functions, a much larger algebra in general.

Consider C^*-algebras, A, B, C, D and *-homomorphisms $\varphi : A \to M(C)$ and $\psi : B \to M(D)$. Suppose that ψ is nondegenerate, then $\varphi \otimes \psi : A \otimes B \to M(C \otimes D)$ extends to $\tilde{M}(A \otimes B)$ and when $\varphi(A) \subset C$, the image of this morphism sits inside $\tilde{M}(C \otimes D)$ and the restricted morphism, denoted again $\varphi \otimes \psi : \tilde{M}(A \otimes B) \to \tilde{M}(C \otimes D)$ is continuous from the B-strict topology of $\tilde{M}(A \otimes B)$ to the D-strict topology of $\tilde{M}(C \otimes D)$.

In particular any *-homomorphism $\varphi : A \to M(B)$ gives rise to a map $\varphi \otimes \mathrm{id}_S : \tilde{M}(A \otimes S) \to M(B \otimes S)$.

In general, we use the same symbol for a $*$-homomorphism and for its extension to the S-multiplier algebras.

Definition 2.2 A Hopf C^*-algebra is a C^*-algebra S endowed with a nondegenerate *-homomorphism $\delta_S : S \to \tilde{M}(S \otimes S)$ which verify the following comultiplication condition

$$(\mathrm{id}_S \otimes \delta_S) \circ \delta_S = (\delta_S \otimes \mathrm{id}_S) \circ \delta_S.$$

A coaction of a Hopf C^*-algebra S on a C^*-algebra A is a nondegenerate *-homomorphism $\delta_A : A \to \tilde{M}(A \otimes S)$ such that

$$(\mathrm{id}_A \otimes \delta_S) \circ \delta_A = (\delta_A \otimes \mathrm{id}_S) \circ \delta_A.$$

This condition is equivalent to the commutativity of the following diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \delta_A\ } & \tilde{M}(A \otimes S) \\
{\scriptstyle \delta_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \delta_S} \\
\tilde{M}(A \otimes S) & \xrightarrow{\ \delta_A \otimes \mathrm{id}_S\ } & M(A \otimes S \otimes S);
\end{array}
$$

we denote by $\delta_A^2 := (\delta_A \otimes \mathrm{id}_S)\delta_A : A \to M(A \otimes S \otimes S)$ the above morphism.

In this case we shall say that A is an S-algebra. An element $a \in A$ is *fixed* by the coaction δ_A if $\delta_A(a) = a \otimes 1$.

Let A and B be S-algebras, a *-homomorphism $\varphi : A \to M(B)$ is S-equivariant (or an S-morphism) if $(\varphi \otimes \mathrm{id}_S)\delta_A = \delta_B \varphi$.

Definition 2.3 A morphism between two Hopf C^*-algebras S and S' is a nondegenerate morphism $\varphi : S \to M(S')$ such that the following diagram commutes

$$
\begin{array}{ccc}
S & \xrightarrow{\;\;\varphi\;\;} & M(S') \\
\delta_S \downarrow & & \downarrow \delta_{S'} \\
\tilde{M}(S \otimes S) & \xrightarrow{\;\varphi \otimes \varphi\;} & M(S' \otimes S').
\end{array}
$$

Such a morphism φ allows us to define an coaction of S' on any S-algebra A by $\delta'_A(a) = (\mathrm{id}_A \otimes \varphi)\delta_A$.

Examples 1. Any C^*-algebra A has a *trivial* Hopf structure with comultiplication defined by $\delta(a) = a \otimes 1$.

2.*Actions of groups.* Let G be a locally compact group (or just a semigroup). Define $\delta : C_0(G) \to C_b(G, C_0(G)) = \tilde{M}(C_0(G) \otimes C_0(G))$ by $\delta(f)(s,t) = f(st)$. Associativity of the product of G translates into the comultiplication property of δ and it is easy to see that $C_0(G)$ is a Hopf C^*-algebra. Strongly continuous actions of G on an algebra A correspond to injective coactions of $C_0(G)$ on A in the following way: let α be an action of G on A, then the corresponding coaction $\delta_A : A \to C_b(G, A)$ is given by $\delta_A(a) = g \mapsto \alpha_g(a)$ for all $a \in A$.

3.*Coactions of groups.* Let G be a locally compact group. The reduced C^*-algebra of G, $C^*_r(G)$, is a Hopf C^*-algebra with comultiplication δ_G given by $\delta(x) = \int x(g)(\lambda_g \otimes \lambda_g)dg$ for an element $x \in C^*_r(G)$, $x = \int x(g)\lambda_g dg$.

Define the unitary W on $L^2(G) \otimes L^2(G) = L^2(G \times G)$ by $Wf(s,t) = f(s, s^{-1}t)$, then the above coproduct can be written as $\delta(x) = W(x \otimes 1)W^*$ from which follows that δ is nondegenerate.

A coaction of this Hopf C^*-algebra is called a coaction of G. If G is commutative, then the Fourier transform provides an isomorphism of Hopf C^*-algebras between $C^*_r(G)$ and $C_0(\hat{G})$. A coaction of G amounts to an action of the dual group.

Let A be a G-algebra, the cross product $A \rtimes_r G$ is endowed with a coaction of G, $\delta_{A \rtimes_r G} : A \rtimes_r G \to \tilde{M}(A \rtimes_r G \otimes C^*_r(G))$ given by $\delta_{A \rtimes_r G}(\int a(s)u_s ds) = \int (a(s) \otimes 1)(u_s \otimes \lambda_s)ds$.

Suppose that A is endowed with a coaction of a *discrete* group G. For every $g \in G$ its spectral subspace is $A_g = \{a \in A \text{ such that } \delta_A(a) = a \otimes \lambda_g\}$, and A is the closed linear span of the family $(A_g)_{g \in G}$. The coaction condition translates into the fact that $(A_g)_{g \in G}$ is a G-grading and to any G-grading there is an associated coaction.

4.*Multiplicative Unitaries.* The operator W is an example of a multiplicative unitary, introduced by Baaj and Skandalis in [2], i.e., an unitary $W \in L(H \otimes H)$ which fulfills the pentagonal relation $W_{12}W_{13}W_{23} = W_{23}W_{12}$. Under supplementary assumptions,

associated to it there are two dual Hopf C^*-algebras and cross product algebras constructions, further extending the results from actions and coactions of groups.

Lemma 2.4 *If the following sequence is exact*

$$0 \to J \otimes S \xrightarrow{i} B \otimes S \xrightarrow{p} A \otimes S \to 0$$

then

$$0 \to \tilde{M}(J \otimes S) \xrightarrow{\bar{i}} \tilde{M}(B \otimes S) \xrightarrow{\bar{p}} \tilde{M}(A \otimes S) \to 0$$

is also exact.

Proof. For the injectivity, i is the inclusion of $J \otimes S$ as an ideal inside $B \otimes S$, $\bar{i}$ is then the inclusion of the corresponding S-completions. Equivalently, $\operatorname{im} \bar{i} = \overline{\operatorname{im} i}^S$, the S-completion of the image of i.

We prove now that $\ker \bar{p} = \overline{\ker p}^S$. Take $b \in \ker \bar{p}$ and s_i an approximate unit for S; the sequence $b_n = b(1 \otimes s_n)$ verifies $b_n \in B \otimes S$, $b_n \in \ker p$ as $\bar{p}(b(1 \otimes s_n)) = \bar{p}(b)(1 \otimes s_n) = 0$, and also $b_n \to b$ strictly, hence $b \in \overline{\ker p}^S$. Thus $\ker \bar{p} \subset \overline{\ker p}^S$. The other inclusion follows from the strict continuity of $\bar{p}$. The initial short exact sequence is exact in the middle thus

$$\operatorname{im} \bar{i} = \overline{\operatorname{im} i}^S = \overline{\ker p}^S = \ker \bar{p}.$$

For the surjectivity of $\bar{p} : \tilde{M}(B \otimes S) \to \tilde{M}(A \otimes S)$ one can adapt the proof of the noncommutative Tietze extension theorem [20, Proposition 6.8], which states that the corresponding map $\tilde{p} : M(B \otimes S) \to M(A \otimes S)$ is surjective. The proof quoted uses the fact the multiplier algebras are strict completions of their corresponding algebras. This proofs adapts to completions in the S-strict topology, that is, to S-multiplier algebras. We outline it here.

First observe that a norm bounded sequence $(x_n)_n$ of elements of $A \otimes S$ converges into the S-strict topology if the sequences $(x_n(1 \otimes h))_n$ and $((1 \otimes h)x_n)_n$ converge in the norm of $A \otimes S$, for some strictly positive element $h \in S$, cf. [27, Lemma 2.3.6.].

Take $x \in \tilde{M}(A \otimes S)$; we are looking for a lift $y \in \tilde{M}(B \otimes S)$. If suffices to take elements x with $0 \leq x \leq 1$; put $x_n := x^{1/2}(1 \otimes s_n)x^{1/2}$, where $(s_n)_n$ denotes an approximate unit for S. One has $(x_n)_n \in A \otimes S$ and $(x_n)_n$ is increasing and converges to x in the S-strict topology.

Choose h a strictly positive element of S; then $((1 \otimes h)x_n(1 \otimes h))_n$ converges in norm to $(1 \otimes h)x(1 \otimes h)$, thus, by eventually passing to a subsequence , we can assume that

$$\|(x_{n+1} - x_n)(1 \otimes h)\| < 2^{-n} \text{ for all } n \text{ and with } x_1 = x_2 = 0.$$

By a lifting argument, there is a sequence $y_n \in B \otimes S$ such that

$$\|y_n\| \leq 1, \ \bar{p}(y_n) = x_n, \text{ and } \|(y_{n+1} - y_n)(1 \otimes h)\| < 2^{-n}.$$

The sequence $(y_n)_n$ converges in the S-strict topology of $\tilde{M}(B \otimes S)$ to an element y such that $\bar{p}(y) = x$. $\square$

Exactness plays a crucial role in the following. Recall that a C^*-algebra D is *exact* if for every short exact sequence of C^*-algebras

$$0 \to J \to B \to A \to 0$$

the sequence associated by taking minimal tensor product with D, i.e., if

$$0 \to J \otimes S \to B \otimes S \to A \otimes S \to 0$$

is also exact.

Related, a group G is *exact*, if the cross product functor $\cdot \rtimes_r G$, is exact, which means that for every short exact sequence of G-algebras $0 \to J \to B \to A \to 0$, the sequence $0 \to J \rtimes_r G \to B \rtimes_r G \to A \rtimes_r G \to 0$ is also exact. In particular, this implies that the reduced C^*-algebra of the group G is exact; in the discrete case this condition is also sufficient ([19]).

Note that if we replace the minimal tensor product with the maximal one, this condition is always satisfied.

We shall consider only coactions of Hopf C^*-algebras S which are exact as C^*-algebras. This framework includes the cases when S is nuclear, like in the case of action by a group, when S is commutative. It also include the case of coactions by an exact group G.

We have chosen to work into the realm of minimal tensor products, but there is also a theory using the maximal one, i.e., when working with coactions given by morphisms $\delta_A : A \to \tilde{M}(A \otimes_{max} S)$. In this case, the exactness condition is satisfied and our theory applies in this case too.

Lemma 2.5 *Let S be an exact C^*-algebra; for every C^*-algebra A and for every $k \geq 1$ there is a natural injective *-homomorphism*

$$i_A^k : \tilde{M}(\mathfrak{A}^k A \otimes S) \to \mathfrak{A}^k \tilde{M}(A \otimes S).$$

Proof. It suffices to consider the case $k = 1$, the general one follows by a similar argument, or by an induction argument. Note first that there is a natural S-strict topology on $C_b(T, A \otimes S)$ defined by the family of seminorms $\|f\|_s = \|f(1 \otimes s)\| + \|(1 \otimes s)f\|$, for all $s \in S$. The natural inclusion map

$$j_A : C_b(T, A) \otimes S \to C_b(T, A \otimes S)$$

is continuous for this S-strict topology. It induces a injective morphism between the corresponding S-completions

$$\bar{j}_A : \tilde{M}(C_b(T, A) \otimes S) \to \overline{C_b(T, A \otimes S)}^S \simeq C_b(T, \tilde{M}(A \otimes S)),$$

the functions on T with values in $\tilde{M}(A \otimes S)$ which are bounded in norm and continuous in the S-strict topology. Moreover, j_A restricts to an isomorphism $C_0(T, A) \otimes S \simeq C_0(T, A \otimes S)$ which in turn induces an isomorphism between the corresponding multiplier algebras, $\tilde{M}(C_0(T, A) \otimes S)$ and $C_0(T, \tilde{M}(A \otimes S))$. Thanks to the exactness of S and the lemma above, we can put these maps into a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{M}(C_0(T, A) \otimes S) & \rightarrowtail & \tilde{M}(C_b(T, A) \otimes S) & \longrightarrow & \tilde{M}(\mathfrak{A}A \otimes S) & \longrightarrow & 0 \\
& & \downarrow{\simeq} & & \downarrow{\bar{j}_A} & & \downarrow{i_A} & & \\
0 & \longrightarrow & C_0(T, \tilde{M}(A \otimes S)) & \rightarrowtail & C_b(T, \tilde{M}(A \otimes S)) & \longrightarrow & \mathfrak{A}\tilde{M}(A \otimes S) & \longrightarrow & 0
\end{array}
$$

and deduce the existence of an injective map:

$$
i_A : \tilde{M}(\mathfrak{A}A \otimes S) \to \mathfrak{A}\tilde{M}(A \otimes S).
$$

Let $\varphi : A \to B$ be a morphism of C^*-algebras. There is a commutative diagram

$$
\begin{array}{ccc}
C_b(T, A) \otimes S & \xrightarrow{(\circ\varphi)\otimes\mathrm{id}_S} & C_b(T, B) \otimes S \\
\downarrow{j_A} & & \downarrow{j_B} \\
C_b(T, A \otimes S) & \xrightarrow{\circ(\varphi\otimes\mathrm{id}_S)} & C_b(T, B \otimes S)
\end{array}
$$

from which follows the commutativity of the following diagram, expressing that the construction of the map i_A is natural:

$$
\begin{array}{ccc}
\tilde{M}(\mathfrak{A}A \otimes S) & \xrightarrow{(\mathfrak{A}\varphi)\otimes\mathrm{id}_S} & \tilde{M}(\mathfrak{A}B \otimes S) \\
\downarrow{i_A} & & \downarrow{i_B} \\
\mathfrak{A}\tilde{M}(A \otimes S) & \xrightarrow{\mathfrak{A}(\varphi\otimes\mathrm{id}_S)} & \mathfrak{A}\tilde{M}(B \otimes S). \quad \square
\end{array}
$$

Having in mind this we can now say what an equivariant asymptotic morphism is:

Definition 2.6 An asymptotic morphism $\varphi : A \to \mathfrak{A}^k B$ is S-equivariant if the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & \mathfrak{A}^k B \\
\downarrow{\delta_A} & & \searrow{\mathfrak{A}^k \delta_B} \\
\tilde{M}(A \otimes S) \xrightarrow{\varphi\otimes\mathrm{id}_S} & \tilde{M}(\mathfrak{A}^k B \otimes S) \xrightarrow{i_B^k} & \mathfrak{A}^k \tilde{M}(B \otimes S)
\end{array}
$$

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Notation. Assume that $(a_t)_{t \in T}$ and $(b_t)_{t \in T}$ are families of elements in a C^*-algebra D. We note $a_t \sim b_t$ if $\lim\limits_{t \longrightarrow +\infty} \|a_t - b_t\| = 0$.

For an asymptotic morphism $\varphi : A \to \mathfrak{A}B$ represented by a family of maps $(\varphi_t)_{t \in T}$, the equivariance condition reads:

$$(\varphi_t \otimes \mathrm{id})(\delta_A(a))(1 \otimes s) \sim \delta_B(\varphi_t(b))(1 \otimes s)$$

for all $s \in S$. Note that the exactness of the algebra S is necessary in defining the asymptotic morphism $\varphi \otimes \mathrm{id}_S$, represented by the family of maps $\varphi_t \otimes \mathrm{id}_S : \tilde{M}(A \otimes S) \to \tilde{M}(B \otimes S)$.

Examples 1. In the case of the Hopf C^*-algebra associated to a group G acting on C^*-algebras A and B, an asymptotic morphism $(\varphi_t)_t : A \to B$ is G-equivariant if

$$\varphi_t(g(a)) \sim g(\varphi_t(a))$$

for all $a \in A$ and uniformly on compacts in $g \in G$, the definition used by Guentner, Higson and Trout in [13].

When the group G is compact, one can average over it and find a representative for the asymptotic morphism φ such that for every $t \in T$ the map $\varphi_t : A \to B$ is equivariant, in the sense that $\varphi_t(g(a)) = g(\varphi_t(a))$. The dual affirmation also holds as follows:

2. In the case of the Hopf C^*-algebra associated to a coaction by a discrete group G, i.e., $S = C_r^*(G)$, an asymptotic morphism $\varphi : A \to \mathfrak{A}B$ is S-equivariant if $\varphi(A_g) \subset \mathfrak{A}B_g$. Let $(\varphi_t)_t : A \to B$ be any representative of φ; define for each $g \in G$ a family $\psi_t^g : A_g \to B_g$ by $\psi_t^g = P_{B_g} \circ \varphi_t$, where $P_{B_g} : B \to B_g$ is the projection on B_g. The family ψ_t^g extends to a family $(\psi_t)_t : A \to B$ which is equivalent to $(\varphi_t)_t : A \to B$ and such that $\psi_t(A_g) \subset B_g$.

It follows from the injectivity of i_B that, if the asymptotic morphism $\varphi : A \to \mathfrak{A}^k B$ is S-equivariant, $\mathfrak{A}^k \delta_B$ restricts to a map

$$\delta_\varphi : \mathrm{im}(\varphi) \to \tilde{M}(\mathrm{im}(\varphi) \otimes S).$$

Consider now $(e_i)_i$ an approximate unit for A, then $\varphi(e_i)$ is an approximate unit for $\mathrm{im}(\varphi)$. $\delta_A(e_i)$ is an approximate unit for $\tilde{M}(A \otimes S)$, so $\delta_\varphi(\varphi(e_i)) = (\varphi \otimes \mathrm{id}_S)(\delta_A(e_i))$ is a approximate unit for $(\varphi \otimes \mathrm{id}_S)(\tilde{M}(A \otimes S)) = \tilde{M}(\mathrm{im}\varphi \otimes S)$, hence δ_φ is nondegenerate. Thus $\mathfrak{A}^k \delta_B$ restricts to an action on the image of an S-equivariant asymptotic morphism.

In the case of an action of a group G this parallels the fact that the image of a G-continuous element is also G-continuous. Hence, even if the action on $\mathfrak{A}B$ is not continuous in general, one can restrict to the subalgebra of G-continuous elements of $\mathfrak{A}B$. We now describe how this works in the case of a coaction of a Hopf C^*-algebra S.

Lemma 2.7 *Let A by a S-algebra. The *-homomorphism $\alpha_A : A \to \mathfrak{A}A$ is S-equivariant.*

Proof. In the diagram expressing the condition that α_A is equivariant, i_A is an isomorphism from $\tilde{M}(\alpha_A(A) \otimes S)$ to the classes of constant functions in $\mathfrak{A}\tilde{M}(A \otimes S)$ and both sides associate to an element $a \in A$ the class of the constant function $t \mapsto \delta_A(a) \in \mathfrak{A}\tilde{M}(A \otimes S)$. $\square$

Definition 2.8 Let A be an S-algebra, define $\mathfrak{A}_S^k A$ the subalgebra of $\mathfrak{A}^k A$ generated by all elements x for which there is a subalgebra $D_x \subset \mathfrak{A}^k A$ with the property that $\mathfrak{A}^k \delta_B$ restricts to a nondegenerate *-homomorphism $\delta_{D_x} : D_x \to \tilde{M}(D_x \otimes S)$.

It follows from the preceding lemma that the algebra $\mathfrak{A}^k A$ is nonzero. The properties of this construction are summarized as follows:

Proposition 2.9 *(i) For every S-algebra A, $\mathfrak{A}_S^k A$ is an S-algebra with comultiplication given by the restriction of $\mathfrak{A}^k \delta_A$ to $\mathfrak{A}_S^k A$.*
(ii) $\mathfrak{A}_S^k$ is a functor from the category of S-algebras and S-equivariant morphisms to itself.
(iii) There is a natural equivariant injection of the k-fold composition of the functor $\mathfrak{A}_S$ with itself into $\mathfrak{A}_S^k$.

Proof.(i) The map $\mathfrak{A}^k \delta_A$ which restricted to each D_x gives an S-algebra structure extends to the C^*-algebra generated by all this subalgebras of $\mathfrak{A}^k A$. To see that this *-homomorphism is nondegenerate observe that $[\mathfrak{A}^k \delta_A(D_x)](D_x \otimes S)$ is dense in $D_x \otimes S$ hence $[\mathfrak{A}^k \delta_A(\mathfrak{A}_S^k A)](\mathfrak{A}_S^k A \otimes S)$ is dense in $D_x \otimes S$ and thus also in $\mathfrak{A}_S^k A \otimes S$.

For (ii) we restrict to the case $k = 1$, the general one works in a similar way. Let $D \subset \mathfrak{A}A$ be an algebra for which $\mathfrak{A}\delta_A$ restricts to an action of S, thus in the following diagram:

the left side commutes. The upper right square is commutative because it is the image by $\mathfrak{A}$ (which is a functor on the category of C^*-algebras) of the square expressing the S-equivariance of φ. Finally, the right lower square commutes by the naturality

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of the applications i_A and i_B. It follows that $\mathfrak{A}\varphi(D) \subset \mathfrak{A}_S B$, which implies that $\mathfrak{A}\varphi(\mathfrak{A}_S A) \subset \mathfrak{A}_S B$, and also that $\mathfrak{A}\varphi$ is S-equivariant.

(iii) It is enough to consider the case $k = 2$; let D be a subalgebra of $\mathfrak{A}(\mathfrak{A}_S A)$ which are among those generating $\mathfrak{A}_S(\mathfrak{A}_S A)$, there is a commutative diagram:

$$
\begin{array}{ccccc}
D & \longrightarrow & \mathfrak{A}(\mathfrak{A}_S A) & \longrightarrow & \mathfrak{A}^2 A \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{M}(D \otimes S) & \longrightarrow & \mathfrak{A}\tilde{M}(\mathfrak{A}_S A \otimes S) & \longrightarrow & \mathfrak{A}^2 \tilde{M}(A \otimes S)
\end{array}
$$

with horizontal maps given by inclusion and vertical maps by the restriction of $\mathfrak{A}^2 \delta_A$. It follows that $D \subset \mathfrak{A}_S^2 A$ and, by the definition of $\mathfrak{A}_S$, that $\mathfrak{A}_S(\mathfrak{A}_S A) \subset \mathfrak{A}_S^2 A$. The equivariance and the naturality of this inclusion follow from the corresponding commutative diagrams. $\square$

Given an S-algebra A, denote by ΣA, the suspension of A, i.e. the tensor product algebra $C_0(\mathbb{R}) \otimes A$, with the trivial coaction of S on $C_0(\mathbb{R})$.

One key ingredient of E-theory is the Connes-Higson construction of an asymptotic morphism which associates to a short exact sequence $0 \to J \to B \to A \to 0$ of separable C^*-algebras an asymptotic morphism $\partial : \Sigma A \to \mathfrak{A} J$ from the suspension of A to J.

We now show that the same construction exists also in the S-equivariant setting.

The Connes-Higson asymptotic morphism is based on the following notion:

Definition 2.10 Let J be an ideal in a C^*-algebra B; a quasicentral approximate unit for $J \lhd B$ is a norm continuous family $(u_t)_{t \in T}$ of elements of J such that
(i) $0 \leq u_t \leq 1$,
(ii) $u_t j \sim j$,
(iii) $u_t b \sim b u_t$,
for all $j \in J$ and $b \in B$.

The Connes-Higson asymptotic morphism is defined as the the asymptotic family which, for $f \in C_0(0, 1)$ and $a \in A$, is given by

$$\partial_t(f \otimes a) = f(u_t)q(a)$$

for some choices of an approximate unit $(u_t)_{t \in T}$ for $J \lhd B$ and of a section $q : A \to B$. It turns out that the homotopy class of this asymptotic morphism do not depend on these choices so the formula defines an element of $\{\Sigma A, J\}$.

We assume now that B is an S-algebra and that the coaction of S on B restricts to a coaction on the ideal J.

Definition 2.11 An S-quasicentral approximate unit for $J \lhd B$ is a quasicentral approximate unit $(u_t)_{t \in T}$ such that

$$\delta(u_t)(1 \otimes s) \sim (u_t \otimes 1)(1 \otimes s)$$

for all $j \in J$, $b \in B$ and $s \in S$.

It was proven in [1, Lemma 4.1] that such approximate units exist provided that B and S are separable. In the case of an action of a group G, this latest condition states that $g(u_t) \sim u_t$, uniformly on the compacts of G.

Theorem 2.12 *Let $0 \to J \to B \to A \to 0$ be an S-extension; the Connes-Higson asymptotic morphism $\partial : \Sigma A \to \mathfrak{A} J$ associated to it is S-equivariant.*

Proof. We have to verify the equivariance condition which in this case is :

$$i_S \circ (\partial \otimes \mathrm{id}_S) \circ \delta_{\Sigma A} = \mathfrak{A} \delta_J \circ \partial$$

in $\mathfrak{A} \tilde{M}(J \otimes S)$ or in an equivalently that $\delta_J(\partial_t(f \otimes a))(1 \otimes s) \sim (\partial_t \otimes \mathrm{id})(\delta_{\Sigma A}(f \otimes a))(1 \otimes s)$ for every $f \in C_0(0,1)$, $a \in A$ and $s \in S$.

The action on Σ is trivial thus

$$(\partial_t \otimes \mathrm{id}_S)(\delta_{\Sigma A}(f \otimes a)) = (\partial_t \otimes \mathrm{id}_S)(f \otimes \delta_A(a)).$$

The family of maps $\partial_t \otimes \mathrm{id}_S : \Sigma A \otimes S \to J \otimes S$ is asymptotically equivalent to the family associated to the short exact sequence:

$$0 \to J \otimes S \to B \otimes S \to A \otimes S \to 0$$

[13, Proposition 5.9], which in turn is given by $f(u_t \otimes v_t)\tilde{q}(\delta_A(a))$, where $(v_t)_{t \in T}$ is an approximate unit for the C^*-algebra S, and $\tilde{q} : A \otimes S \to B \otimes S$ is a section.

Note that

$$f(u_t \otimes v_t)\tilde{q}(\delta_A(a)) \sim f(u_t \otimes v_t)\delta_B(q(a))$$

as $\tilde{q}(\delta_A(a)) - \delta_B(q(a)) \in \ker(p \otimes \mathrm{id}_S) = J \otimes S$ and $f(u_t \otimes v_t)h \sim 0$ for all $h \in J \otimes S$ (see [13, Lemma 5.6]). $f(u_t \otimes v_t) \sim f(u_t \otimes 1)$ in the S-strict topology of $\tilde{M}(J \otimes S)$, as the condition $\|[f(u_t \otimes v_t) - f(u_t \otimes 1)](1 \otimes s)\| \to 0$ is easily verified for $f(x) = x$ and by the Weierstrass approximation theorem holds for all $f \in C_0(0,1)$. Consequently

$$(\partial_t \otimes \mathrm{id}_S)(\delta_{\Sigma A}(f \otimes a))(1 \otimes s) \sim f(u_t \otimes 1)\delta_B(q(a))(1 \otimes s),$$

where we regard $\delta_B(q(a)) \in \tilde{M}(B \otimes S)$ as an element of $M(B \otimes S)$ and, by restriction, as an element of $M(J \otimes S)$.

On the other hand, $\delta_J(\partial_t(f \otimes a)) = f(\delta_J(u_t))\delta_B(q(a))$, hence we have to prove that $f(\delta_J(u_t)) \sim f(u_t \otimes 1)$ in the strict topology of $\tilde{M}(J \otimes S)$ which follows again

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from Weierstrass's theorem and the S-equivariance of $(u_t)_{t \in T}$. $\square$

Like in the non-equivariant case, the homotopy class of the Connes-Higson asymptotic morphism does not depend on the choices of the approximate unit u_t and of the section q.

It is worth mentioning that, up to suspension, every asymptotic morphism is obtained in this way from an extension, hence the theorem shows that the definition of an S-asymptotic morphism is appropriate.

3   E-theory

In the nonequivariant case, E-theory groups are defined as

$$E(A, B) = \{\Sigma A \otimes \mathcal{K}, \Sigma B \otimes \mathcal{K}\},$$

the homotopy classes of asymptotic morphisms between the algebras $\Sigma A \otimes \mathcal{K}$ and $\Sigma B \otimes \mathcal{K}$. In the case of a coaction by a Hopf algebra S, a similar definition of E^S, using S-asymptotic morphisms instead of nonequivariant ones, would have many good properties, but not a crucial one: to insure the existence of a natural transformation, $KK^S \to E^S$, from equivariant KK-theory to it. In order to have such a property, one has to define E^S in such a way that it has the same stability properties as KK^S.

In the case of actions by a group G, one can take advantage of the the G-module $L^2(G)$ endowed with the left regular representation. It has the following property ([22, Lemma 2.3]): let A be a G-algebra, and let E_1 and E_2 be Hilbert G-A-modules which are (non-equivariantly) isomorphic as Hilbert A-modules, then $L^2(G, E_1)$ and $L^2(G, E_2)$ are isomorphic Hilbert G-A-modules.

Here a module $L^2(G, E)$ denotes the tensor product of the modules $L^2(G)$ and E.

Denote by $\mathcal{K}_G$ the G-algebra of compact operators of the module $L^2(G)^\infty$, with the action of G induced from the action on $L^2(G)$, denoted by λ.

In the case of an action of a group G, the equivariant E-theory, E^G, is defined in [13] as

$$E^G(A, B) = \{\Sigma A \otimes \mathcal{K}_G, \Sigma B \otimes \mathcal{K}_G\}.$$

As explained in the appendix, this insures equivariant stability.

In the more general case of a coaction by a Hopf C^*-algebra S, we do not have a substitute for the module $L^2(G)$.

To solve this problem, we construct a category of S-algebras in which we allow a bit more freedom for the coaction on the algebras $A \otimes \mathcal{K}$ besides the coaction $\delta_A \otimes \mathrm{id}_{\mathcal{K}}$, in the sense of the following definition. We shall show in the next section that this is exactly what we need.

Definition 3.1 Let (S, δ_S) be a Hopf-C^*-algebra, and (A, δ_A) be an (S, δ_S)-algebra. A coaction $\delta_{A \otimes \mathcal{K}} : A \otimes \mathcal{K} \to \tilde{M}(A \otimes \mathcal{K} \otimes S)$ of S on $A \otimes \mathcal{K}$ is compatible with the

coaction δ_A of S on A if there is a minimal projection $e \in \mathcal{K}$ such that the map $a \to a \otimes e$ is equivariant.

A key notion is the following:

Definition 3.2 Let (S, δ_S) be a Hopf-C^*-algebra, and (A, δ_A) be an (S, δ_S)-algebra. A δ_A-cocycle is an unitary $V \in M(A \otimes S)$ such that

$$(\mathrm{id}_A \otimes \delta_S)V = (V \otimes \mathrm{id}_S)[(\delta_A \otimes \mathrm{id}_S)(V)].$$

Example. Let G be a locally compact group and $C_0(G)$ the associated Hopf algebra. A δ_A-cocyle is a strongly continuous map from G into the unitary group of A such that $V_{g_1 g_2} = V_{g_1} \alpha_{g_1}(V_{g_2})$. (cf. [4, Def. 2.2.3])

Lemma 3.3 If V is a δ_A-cocyle, then $\delta_A^V(\cdot) := V\delta_A(\cdot)V^*$ is a coaction of S on A.

Proof. The S-strict topology is stronger than the strict topology induced from $M(A \otimes S)$, so comparing completions yields an identification $M(\tilde{M}(A \otimes S)) = M(A \otimes S)$. Hence $\delta_A^V(\cdot) : A \to \tilde{M}(A \otimes \mathcal{K})$ is a nondegenerate map. The equality $(\mathrm{id}_A \otimes \delta_S)\delta_A^V(\cdot) = (\delta_A^V \otimes \mathrm{id}_S)\delta_A^V(\cdot)$ is easy to check. $\square$

Note that if $\varphi : (A, \delta_A) \to (B, \delta_B)$ is a surjective S-morphism and V is a δ_A-cocycle, then $(\varphi \otimes \mathrm{id}_S)(V)$ is a δ_B-cocycle.

The following lemma is a generalization of a result due to Connes in the case of a group acting on a C^*-algebra.

Lemma 3.4 Let (A, δ_A) be an S-algebra; there is a coaction $\delta_{M_2(A)} : M_2(A) \to \tilde{M}(M_2(A) \otimes S)$ of S on $M_2(A)$ with the property that for all $a, b \in A$,

$$\delta_{M_2(A)} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \delta_A(a) & 0 \\ 0 & \delta_A'(b) \end{pmatrix},$$

for some coaction δ_A' of S on A, if and only if there is a δ_A-cocycle $V \in M(A \otimes S)$ such that $\delta_{M_2(A)}$ is given by $\delta_{M_2(A)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta_A(a) & \delta_A(b)V^* \\ V\delta_A(c) & V\delta_A(d)V^* \end{pmatrix}$.

Proof. The converse is obvious. Let $\begin{pmatrix} A & B \\ V & D \end{pmatrix}$ be $\delta_{M_2(A)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. It follows easily that $A = B = D = 0$, and that V is an unitary in $M(A \otimes S)$.

$$
\begin{aligned}
(\mathrm{id}_{M_2(A)} \otimes \delta_S) \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} &= (\mathrm{id}_{M_2(A)} \otimes \delta_S)\delta_{M_2(A)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= (\delta_{M_2(A)} \otimes \mathrm{id}_S)\delta_{M_2(A)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned}
$$

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$$= (\delta_{M_2(A)} \otimes \mathrm{id}_S) \begin{pmatrix} 0 & 0 \\ 1_{A \otimes S} & 0 \end{pmatrix} (\delta_{M_2(A)} \otimes \mathrm{id}_S) \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \otimes \mathrm{id}_S \begin{pmatrix} (\delta_A \otimes \mathrm{id}_S)V & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ (V \otimes \mathrm{id}_S)(\delta_A \otimes \mathrm{id}_S)V & 0 \end{pmatrix},$$

which proves that V is an δ_A-cocycle. $\square$

The coactions δ_A and δ'_A like in the lemma above are called *exterior equivalent*.

An advantage of working with cocycles is that, given a coaction δ_A of S on A, we can control a compatible coaction on $A \otimes \mathcal{K}$. This goes as follows: let $\delta_{A \otimes \mathcal{K}} : A \otimes \mathcal{K} \to \tilde{M}(A \otimes \mathcal{K} \otimes S)$ be such a compatible action and let $\{e_{ij} | \ i, j \geq 1\}$ be a set of matrix units for $\mathcal{K}$ arranged such that $e_{11} = e$; define

$$V = \sum (1_A \otimes e_{i1} \otimes 1_S) \delta_{A \otimes \mathcal{K}}(1 \otimes e_{1i}).$$

Lemma 3.5 V *is a* $\delta_{A \otimes \mathcal{K}}$-cocycle and $V \delta_{A \otimes \mathcal{K}}(\cdot) V^* = \delta_A(\cdot) \otimes \mathrm{id}_{\mathcal{K}}$.

Proof. It is easy to see that V is an unitary; let us check the cocycle condition:

$$(V \otimes \mathrm{id}_S)[(\delta_{A \otimes \mathcal{K}} \otimes \mathrm{id}_S)(V)] = \sum_{i,j} [1_A \otimes e_{i1} \otimes 1_{S \otimes S}][\delta_{A \otimes \mathcal{K}}(1_A \otimes e_{1i}) \otimes 1_S][\delta_{A \otimes \mathcal{K}}(1_A \otimes e_{j1}) \otimes 1_S]$$

$$[(\delta_{A \otimes \mathcal{K}} \otimes \mathrm{id}_S)\delta_{A \otimes \mathcal{K}}(1 \otimes e_{j1})]$$

$$= \sum_{i,j} [1_A \otimes e_{i1} \otimes 1_{S \otimes S}][\delta_{A \otimes \mathcal{K}}((1_A \otimes e_{1i})(1_A \otimes e_{j1})) \otimes 1_S]$$

$$[(\mathrm{id}_{A \otimes \mathcal{K}} \otimes \delta_S)\delta_{A \otimes \mathcal{K}}(1 \otimes e_{j1})]$$

$$= \sum_{i} [1_A \otimes e_{i1} \otimes 1_{S \otimes S}][\delta_{A \otimes \mathcal{K}}(1_A \otimes e_{11}) \otimes 1_S]$$

$$[(\mathrm{id}_{A \otimes \mathcal{K}} \otimes \delta_S)\delta_{A \otimes \mathcal{K}}(1 \otimes e_{i1})]$$

$$= \sum_{i} [1_A \otimes e_{i1} \otimes 1_{S \otimes S}][1_{A \otimes S} \otimes e_{11} \otimes 1_{S \otimes S}][(\mathrm{id}_{A \otimes \mathcal{K}} \otimes \delta_S)\delta_{A \otimes \mathcal{K}}(1 \otimes e_{i1})]$$

$$= \sum_{i} [1_A \otimes e_{i1} \otimes 1_{S \otimes S}][1_A \otimes e_{11} \otimes 1_S][(\mathrm{id}_{A \otimes \mathcal{K}} \otimes \delta_S)\delta_{A \otimes \mathcal{K}}(1 \otimes e_{i1})]$$

$$= (\mathrm{id}_{A \otimes \mathcal{K}} \otimes \delta_S)(\sum_{i} (1_A \otimes e_{i1} \otimes 1_S)\delta_{A \otimes \mathcal{K}}(1_A \otimes e_{i1} \otimes 1_{S \otimes S}))$$

$$= (\mathrm{id}_{A \otimes \mathcal{K}} \otimes \delta_S)V.$$

Also, for every $p, q \geq 1$:

$$V \delta_{A \otimes \mathcal{K}}(a \otimes e_{pq})V^* = \sum_{i,j} (1_A \otimes e_{i1} \otimes 1_S)\delta_{A \otimes \mathcal{K}}(1_A \otimes e_{1i})\delta_{A \otimes \mathcal{K}}(1_A \otimes e_{pq})\delta_{A \otimes \mathcal{K}}(1_A \otimes e_{j1})$$

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$$(1_A \otimes e_{1j} \otimes 1_S)$$
$$= (1_A \otimes e_{p1} \otimes 1_S)\delta_{A \otimes \mathcal{K}}(1_A \otimes e_{11})(1_A \otimes e_{1q} \otimes 1_S)$$
$$= (1_A \otimes e_{p1} \otimes 1_S)(\delta_A(1_A) \otimes e_{11})(1_A \otimes e_{1q} \otimes 1_S)$$
$$= (\delta_A \otimes \mathrm{id}_{\mathcal{K}})(a \otimes e_{pq}),$$

which proves that $V\delta_A(\cdot)V^* = \delta_A(\cdot) \otimes \mathrm{id}_{\mathcal{K}}$. $\square$

Definition 3.6 Let (A, δ_A) and (B, δ_B) be S-algebras, we call $\bar{S}$-morphism an S-equivariant morphism $\varphi : (A \otimes \mathcal{K}, \delta_A \otimes \mathrm{id}_{\mathcal{K}}) \to (B \otimes \mathcal{K}, \delta_{B \otimes \mathcal{K}})$, where $\delta_{B \otimes \mathcal{K}}$ is a coaction of S on $B \otimes \mathcal{K}$ which is compatible with δ_B.

For an interval $I = [a, b]$ and an S-algebra A, denote by

$$IA = \{f : I \to A \mid f \text{ is continuous}\},$$

the algebra of continuous functions from I to A, regraded as an S-algebra with the trivial coaction of S on I.

Definition 3.7 Let (A, δ_A) and (B, δ_B) be S-algebras, two $\bar{S}$-morphisms $\varphi_0, \varphi_1 : (A \otimes \mathcal{K}, \delta_A \otimes \mathrm{id}_{\mathcal{K}}) \to (B \otimes \mathcal{K}, \delta_{B \otimes \mathcal{K}})$ are S-homotopic, if there is an S-morphism $\Phi : (A \otimes \mathcal{K}, \delta_A \otimes \mathrm{id}_{\mathcal{K}}) \to (I(B \otimes \mathcal{K}), \mathrm{id}_I \otimes \delta_{B \otimes \mathcal{K}})$ such that φ_0 and φ_1 are obtained by evaluation of Φ at the endpoints of the interval I.

We shall write $\varphi_0 \sim_{\bar{S}} \varphi_1$ in this case; S-homotopy is an equivalence relation.

Recall that a morphism $\varphi : A \to B$ is *quasiunital* if there is an approximate unit $(e_i)_i$ for A such that $(\varphi(e_i))_i$ converges strictly to a projection $p \in M(B)$. This condition is equivalent to the existence of an extension of φ to a strictly continuous map $\bar{\varphi} : M(A) \to M(B)$ between the multiplier algebras.

We say that an S-algebra (A, δ_A) is *trivially stable*, if there is an S-equivariant isomorphism $(A, \delta_A) \simeq (A \otimes \mathcal{K}, \delta_A \otimes \mathrm{id}_{\mathcal{K}})$.

Lemma 3.8 *Let (A, δ_A), (B, δ_B) be S-algebras, and assume that B is trivially stable, then every $\bar{S}$-morphism between A and B is $\bar{S}$-homotopic to a quasiunital one.*

Proof. In the non equivariant case this is [16, Theorem 1.3.16]. In our case, take $\varphi : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ to be a $*$-homomorphism, $\delta_A \otimes \mathrm{id}_{\mathcal{K}}$-$\delta_{B \otimes \mathcal{K}}$-equivariant for some coaction $\delta_{B \otimes \mathcal{K}}$ on $B \otimes \mathcal{K}$. Because B is trivially stable and $\delta_{B \otimes \mathcal{K}}$ is compatible with δ_B, it follows that

$$(B \otimes \mathcal{K}, \delta_{B \otimes \mathcal{K}}) \simeq ((B \otimes \mathcal{K}) \otimes \mathcal{K}, \delta_{B \otimes \mathcal{K}} \otimes \mathrm{id}_{\mathcal{K}}).$$

One can check now that the decompositions in the proof of the theorem just quoted carry over to our case, as the action on the last copy of $\mathcal{K}$ is trivial. $\square$

From now on, we assume in this section that all algebras are trivially stable. This is not a restriction as, for our purposes, one can replace any algebra by its tensor product by the compacts, and have a trivially stable one. Thus when we write $A \otimes \mathcal{K}$ with some compatible coaction $\delta_{A \otimes \mathcal{K}}$, we imply that the algebra A is also trivially stable, but we do not specify an extra copy of $\mathcal{K}$.

Proposition 3.9 *There is a category, denoted $\bar{S}-$alg, whose objects are S-algebras and whose morphisms are S-homotopy classes of $\bar{S}$-morphisms.*

Proof. First suppose that $\varphi : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ is a quasi-unital $*$-homomorphism which is $\delta_{A \otimes \mathcal{K}}$-$\delta_{B \otimes \mathcal{K}}$-equivariant, for a coaction $\delta_{A \otimes \mathcal{K}}$ on $A \otimes \mathcal{K}$ compatible with the given coaction δ_A on A, and a coaction $\delta_{B \otimes \mathcal{K}}$ on $B \otimes \mathcal{K}$, compatible with the given coaction δ_B on B. Then φ is among our $\bar{S}$-morphisms as follows.

Denote by $p := \varphi(1)$; p is a projection in the the center of the multipliers $M(B \otimes \mathcal{K})$ and $B \otimes \mathcal{K}$ decomposes as a direct sum of algebras $p(B \otimes \mathcal{K}) \oplus (1-p)(B \otimes \mathcal{K})$. Moreover, p and $1 - p$ are $\delta_{B \otimes \mathcal{K}}$-equivariant and there is an identification

$$(p \otimes 1_S) \tilde{M}(B \otimes \mathcal{K} \otimes S) = \tilde{M}(p(B \otimes \mathcal{K}) \otimes S)$$

which is the S-completion of the identification $(p \otimes 1_S)(B \otimes \mathcal{K} \otimes S) = p(B \otimes \mathcal{K}) \otimes S$. It follows that the action $\delta_{B \otimes \mathcal{K}}$ decomposes as a sum

$$p\delta_{B \otimes \mathcal{K}} \oplus (1-p)\delta_{B \otimes \mathcal{K}} : p(B \otimes \mathcal{K}) \oplus (1-p)(B \otimes \mathcal{K}) \to \tilde{M}(p(B \otimes \mathcal{K}) \otimes S) \oplus \tilde{M}((1-p)(B \otimes \mathcal{K}) \otimes S).$$

Denote by $U \in M(B \otimes \mathcal{K} \otimes S)$ the $\delta_{A \otimes \mathcal{K}}$-cocycle for which $\delta_{A \otimes \mathcal{K}}(\cdot) = U(\delta_A \otimes \mathrm{id}_{\mathcal{K}})(\cdot)U^*$; using the decomposition above we denote by $\varphi^{\sharp}(U)$ the unitary $(\varphi \otimes \mathrm{id}_S)(U) \oplus 1$. It follows that $\varphi^{\sharp}(U)$ is a $\delta_{B \otimes \mathcal{K}}$-cocycle and that φ is $\delta_A \otimes \mathrm{id}_{\mathcal{K}}(\cdot)$-$\varphi^{\sharp}(U)\delta_{B \otimes \mathcal{K}}(\cdot)\varphi^{\sharp}(U)$-equivariant, which proves the claim.

For the composition, let $\varphi : (A, \delta_A \otimes \mathrm{id}_{\mathcal{K}}) \to (B, \delta_{B \otimes \mathcal{K}})$ and $\psi : (B, \delta_B \otimes \mathrm{id}_{\mathcal{K}}) \to (C, \delta_{C \otimes \mathcal{K}})$ be two $\bar{S}$-morphisms. We can write $\delta_{B \otimes \mathcal{K}}(\cdot) = U_B(\delta_B \otimes \mathrm{id}_K)(\cdot)U_B^*$ for some $\delta_{B \otimes \mathcal{K}}$-cocycle U_B. If follows that the morphism $\psi \circ \varphi$ is $(\delta_A \otimes \mathrm{id}_{\mathcal{K}})(\cdot)$- $\psi^{\sharp}(U_B)(\delta_C \otimes \mathrm{id}_{\mathcal{K}})(\cdot)(\psi^{\sharp}(U_B))^*$-equivariant, which concludes the proof. $\square$

Denote by $\{A, B\}_0^{\bar{S}}$ the homotopy classes of $\bar{S}$-morphisms between A and B, i.e., the morphisms in $\bar{S}-$alg between A and B.

Definition 3.10 Let A, B be two S-algebras. An $\bar{S}$-asymptotic morphism is an S-morphism $\varphi : A \to \mathfrak{A}_S^n B$, for some $n \geq 1$.

Let (A, δ_A) and (B, δ_B) be S-algebras and take two $\bar{S}$-asymptotic morphisms between them, φ_0 and φ_1; we say that they are n-S-homotopic, if there is an S-morphism $\Phi : (A \otimes \mathcal{K}, \delta_A \otimes \mathrm{id}_{\mathcal{K}}) \to \mathfrak{A}_S^n((I(B \otimes \mathcal{K})), \mathrm{id}_I \otimes \delta_{\mathfrak{A}_S^n(B \otimes \mathcal{K})})$ such that φ_0 and φ_1 are obtained by evaluation of Φ at the endpoints of the interval I, for some compatible coaction $\delta_{\mathfrak{A}_S^n(B \otimes \mathcal{K})}$ on $B \otimes \mathcal{K}$.

It follows like in the group case, [13, Proposition 2.3], that

Lemma 3.11 *n-homotopy is an equivalence relation on $\bar{S}$-asymptotic morphisms from A to $\mathfrak{A}_S^n B$.*

Denote by $\{A, B\}_n^{\bar{S}}$ the set of n-homotopy classes of $\bar{S}$-asymptotic morphisms from A to $\mathfrak{A}_S^n B$.

Given an $\bar{S}$-asymptotic morphism $\varphi : A \longrightarrow \mathfrak{A}_S^n B$, the composition

$$A \xrightarrow{\varphi} \mathfrak{A}_S^n B \xrightarrow{\mathfrak{A}^n(\alpha_B)} \mathfrak{A}_S^{n+1} B$$

is an $\bar{S}$-asymptotic morphism; this map agrees with n-S-homotopy of asymptotic morphisms and thus induces a map $\{A, B\}_n^{\bar{S}} \to \{A, B\}_{n+1}^{\bar{S}}$; an equivalent way of defining this map is $A \xrightarrow{\varphi} \mathfrak{A}_S^n B \xrightarrow{\alpha_{\mathfrak{A}^n B}} \mathfrak{A}_S^{n+1} B$.

Definition 3.12 Denote by $\{A, B\}^{\bar{S}}$ the inductive limit $\varinjlim \{A, B\}_n^{\bar{S}}$.

Theorem 3.13 *Let A, B, and C be S-algebras. Given $\bar{S}$-asymptotic morphisms $\varphi : A \to \mathfrak{A}_S^j B$ and $\psi : B \to \mathfrak{A}_S^k C$, the formula $A \xrightarrow{\varphi} \mathfrak{A}_S^j B \xrightarrow{\mathfrak{A}^j(\psi)} \mathfrak{A}_S^{j+k} C$, defines an associative composition law*

$$\{A, B\}^{\bar{S}} \times \{B, C\}^{\bar{S}} \longrightarrow \{A, C\}^{\bar{S}}.$$

Moreover, if A is separable, the inclusion $\{A, B\}_1^{\bar{S}} \to \{A, B\}^{\bar{S}}$ is a bijection.

Proof. The proof of the first part is exactly the same as in the group case [13, Proposition 2.12]: the homotopies used in the proof can be used in the S-equivariant case.

The second part is an alternative formulation of the composition of asymptotic morphisms given in [9], and adapts as follows.

The key ingredient is the following result: let $D \subset \mathfrak{A}^2 C$ be a separable algebra, there exists a function $r : T \to T$ such that the restriction of a two variables function to the graph of r defines a *-homomorphism $R : D \to \mathfrak{A}C$. Moreover, the inclusion $D \subset \mathfrak{A}^2 C$ is 2-homotopic to the composition $\alpha_C \circ R$ and hence factorizes through $\mathfrak{A}C$. This is the first part of [13, Lemma 2.17]. The second part states that this choice of the function r can be done with respect to a map between separable subalgebras $D_1 \to D_2$. The S-equivariant case follows from this by taking $D_2 = \delta_{D_1}(D_1) \subset \tilde{M}(D_1 \otimes S)$.

The 2-homotopy used in this factorization is given by

$$H(t_1, t_2, s) = \begin{cases} F(t_1, t_2) & \text{if } t_1 > sr(t_2) \\ F(sr(t_2), t_2) & \text{if } t_1 \le sr(t_2) \end{cases}$$

and it is an S-homotopy. $\square$

We can now define the E-theory groups.

Definition 3.14 Let A and B be S-algebras; we define the equivariant E-theory of A and B as

$$E^S(A, B) = \{\Sigma A, \Sigma B\}^{\bar{S}}.$$

An asymptotic morphism $\varphi_t : \Sigma A \otimes \mathcal{K} \to \Sigma B \otimes \mathcal{K}$ for which there is a norm continuous family of $\delta_B \otimes \mathrm{id}_{\Sigma \otimes \mathcal{K}}$-cocycles $U_t \in M(\Sigma B \otimes \mathcal{K} \otimes S)$ such that

$$(\varphi_t \otimes \mathrm{id}_S)(\delta_A(a)) \sim U_t \delta_B(\varphi_t(a)) U_t^*$$

in the S-strict topology will define an element of $E^S(A, B)$.

Example. Let us take a closer look at the case the group case; assume that the C^*-algebras A and B are endowed with coaction of a group G, denoted respectively with α, and β.

Let $\varphi_t : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ be an asymptotic morphism, and assume that

$$\varphi_t(\alpha_g(a)) \sim \beta_g^t(\varphi_t(a)),$$

where β^t is a continuous family of actions on $B \otimes \mathcal{K}$, all exterior equivalent with the given action β. Such an element defines an element of the E-theory group $E^{C_0(G)}(A, B)$.

An important example of such an asymptotic morphism appears in the proof of the Baum-Connes for amenable groups where the family β^t is due to a deformation of the action on a Hilbert space H, reflected on the algebra of compact operators $\mathcal{K} = \mathcal{K}(H)$ (see [12, Remark 2.7.6, Proposition 4.6.25]).

As in the nonequivariant case, $E^S(A, B)$ is an abelian group. The sum is defined using an isomorphism $M_2(\mathcal{K}) \simeq \mathcal{K}$ and the inverse is defined using the fact that an element of the type $t \mapsto \begin{pmatrix} f(t) & 0 \\ 0 & f(-t) \end{pmatrix}$ from $C_0(\mathbb{R})$ to a C^*-algebra is null homotopic.

From Theorem 3.13 follows that:

Proposition 3.15 *For any separable S-algebras A, B and C, there is a bilinear composition law*

$$E^S(A, B) \times E^S(B, C) \to E^S(A, C)$$

extending the composition of the category $\bar{S}-$alg. $\square$

Any morphism of Hopf-C^*-algebras $\varphi : S \to M(S')$ induces a restriction natural transformation $E^S(A, B) \to E^{S'}(A, B)$.

It follows form the discussion the the appendix that for every S-algebra A, the functors $E^S(A, \cdot)$ and $E^S(\cdot, A)$ are stable, in the equivariant setting.

E-theory appeared as an answer to the question whether there are always six-terms exact sequences in KK-theory. The key property in establishing this is the following

notion: a functor F from a category of algebras to abelian groups is *half-exact* if given a short exact sequence

$$0 \to J \to B \to A \to 0$$

of algebras, the sequence $F(J) \to F(B) \to F(A)$ is exact. We saw that the Connes-Higson asymptotic morphism is S-equivariant, the proof from the nonequivariant case adapts to show that given an S-algebra D, the functors $E^S(D, \cdot)$ and $E^S(\cdot, D)$ are half-exact. Applying the Puppe exact sequences techniques along with Cuntz' theorem on Bott periodicity yields:

Proposition 3.16 *Let D be an S-algebra and let*

$$0 \to J \to B \to A \to 0$$

be an S-extension. There are six-terms exact sequences:

$$
\begin{array}{ccc}
E^S(D, SJ) \longrightarrow E^S(D, SB) \longrightarrow E^S(D, SA) \\
\uparrow \qquad\qquad\qquad\qquad\qquad \downarrow \\
E^S(D, A) \longleftarrow E^S(D, B) \longleftarrow E^S(D, J)
\end{array}
$$

and

$$
\begin{array}{ccc}
E^S(SJ, D) \longleftarrow E^S(SB, D) \longleftarrow E^S(SA, D) \\
\downarrow \qquad\qquad\qquad\qquad\qquad \uparrow \\
E^S(A, D) \longrightarrow E^S(B, D) \longrightarrow E^S(J, D). \square
\end{array}
$$

Proposition 3.17 *For any exact S-algebra D there is a natural transformation*

$$\tau_D : E^S(A, B) \to E^S(A \otimes D, B \otimes D)$$

such that $\tau_D(1_A) = \tau_{A \otimes D}$.

   Proof. It is shown in [13, Proposition 4.4], that, thanks to the exactness of D, there is a natural map $i_D : \mathfrak{A}^k(A \otimes D) \to \mathfrak{A}^k A \otimes D$ for all $k \geq 0$. The argument is similar to the one used in the proof of the Lemma 2.5 and thus it is easy to see that this map restricts to a map

$$i_D^S : \mathfrak{A}_S^k(A \otimes D) \to \mathfrak{A}_S^k A \otimes D.$$

Composition with i_D^S gives a well defined map

$$\tau_D : \{A, B\}^{\bar{S}} \to \{A \otimes D, B \otimes D\}^{\bar{S}},$$

because the tensor product of two compatible coactions is a compatible coaction. and taking suspensions one obtains a map $\tau_D : E^S(A, B) \to E^S(A \otimes D, B \otimes D)$. The second condition is obvious by the definition of i_D. $\square$

Remark 3.18 Based on this, one we can define

$$\otimes : E^S(A, B) \times E^S(C, D) \to E^S(A \otimes C, B \otimes D)$$

by $\varphi \otimes \psi := (\varphi \otimes 1_C) \circ (1_B \otimes \psi)$ for $\varphi \in E^S(A, B)$ and $\psi \in E^S(C, D)$.

Proposition 3.19 *For every group G and every G-algebras A, B, the E-theory of A and B with respect to coactions of $C_0(G)$, $E^{C_0(G)}(A, B)$, is isomorphic to the E-theory with respect to actions of G, $E^G(A, B)$.*

Proof. We already observed that $C_0(G)$-asymptotic morphisms correspond to G-asymptotic morphisms as defined in [13]. Combining this with the stability of $E^{C_0(G)}$, allows us to replace in the definition of $E^{C_0(G)}(A, B)$, (A, α) with $(A \otimes \mathcal{K}_G, \alpha \otimes \lambda_G)$, and $(\mathfrak{A}_S B \otimes \mathcal{K}, \delta_{\mathfrak{A}_S B \otimes \mathcal{K}})$ with $(\mathfrak{A}_S B \otimes \mathcal{K} \otimes \mathcal{K}_G, \delta_{\mathfrak{A}_S B \otimes \mathcal{K}} \otimes \lambda_G)$. As proven in the appendix, this latest algebra is G-isomorphic to $(\mathfrak{A}_S B \otimes \mathcal{K}_G, \mathfrak{A} \delta_B \otimes \lambda_G)$, from which the result follows. $\square$

We now prove the Baaj-Skandalis isomorphism in E-theory. First let us remind some definitions and properties of the cross products by actions and by coactions of groups on C^*-algebras.

Let A be a C^*-algebra endowed with a coaction $\delta_A : A \to \tilde{M}(A \otimes C_r^*(G))$ of a group G; to construct the cross product algebra with make the assumption that $\delta_A(A)(A \otimes C_r^*(G))$ is dense in $A \otimes C_r^*(G)$, and we say that A is a $\hat{G}$-algebra.

This condition is always fulfilled for coactions of discrete and of amenable groups.

Definition 3.20 Let (A, δ_A) be a $\hat{G}$-algebra; the reduced cross product by the coaction δ_A is the subalgebra of $M(A \otimes \mathcal{K}(L^2(G)))$ spanned by the products $\delta_A(a)(\mathrm{id}_A \otimes M_f)$, for all $a \in A$ and $f \in C_0(G)$, M_f denoting the multiplication operator by f.

We denote by $A \rtimes_r \hat{G}$ this algebra. There are inclusion maps $i_A : A \to \tilde{M}(A \rtimes_r \hat{G})$ given by $i_A(a) := \delta_A(a)$, and $i_G : C_0(G) \to \tilde{M}(A \rtimes_r \hat{G})$ given by $i_G(f) := \mathrm{id}_A \otimes M_f$; the cross product is the closed linear span of the products $i_A(a) i_G(f)$ for $a \in A$ and $f \in C_0(G)$.

The algebra $A \rtimes_r \hat{G}$ is endowed with an action of G, given by $\alpha_g(i_A(a)i_G(f)) = i_A(a)i_G(gf)$, where $g(f)(h) := f(gh)$.

The cross product by a coaction is a functor from the category of $\hat{G}$-algebras to the category of G-algebras: if A and B are $\hat{G}$-algebras and if $f : A \to B$ is an equivariant $*$-homomorphism, then there is an induced morphism $\varphi \times 1 : A \rtimes_r \hat{G} \to B \rtimes_r \hat{G}$, which is given by $\varphi \times 1(i_A(a)i_G(f)) = i_B(\varphi(a))i_G(f)$.

We can now state the analogue of the Takesaki-Takai duality for noncommutative groups.

Let A be a G-algebra, the cross product $A \rtimes_r G$ is a $\hat{G}$-algebra, and there is a G-equivariant isomorphism between $A \rtimes_r G \rtimes_r \hat{G}$ and $A \otimes \mathcal{K}(L^2(G))$.

Similarly, if (A, δ_A) is a $\hat{G}$-algebra, there is a $\hat{G}$-equivariant isomorphism between the double cross product $A \rtimes_r \hat{G} \rtimes_r G$ and $A \otimes \mathcal{K}(L^2(G))$. Here the coaction of G on $A \otimes \mathcal{K}(L^2(G))$ is given by $(\mathrm{id}_A \otimes W^*)(\mathrm{id}_A \otimes \sigma)(\delta_A \otimes \mathrm{id}_{\mathcal{K}(L^2(G))})(\cdot)(\mathrm{id}_A \otimes W)$, with $\sigma : C_r^*(G) \otimes \mathcal{K}(L^2(G)) \to \mathcal{K}(L^2(G)) \otimes C_r^*(G)$ denoting the swap $\sigma(x \otimes y) = y \otimes x$, and with W denoting the unitary $Wf(s,t) = f(s, s^{-1}t)$.

Both for actions and for coactions, under this duality isomorphisms, an equivariant $*$-homomorphism $\varphi : A \to B$ becomes the equivariant $*$-homomorphism $\varphi \otimes \mathrm{id}_{\mathcal{K}(L^2(G))} : A \otimes \mathcal{K}(L^2(G)) \to B \otimes \mathcal{K}(L^2(G))$.

Let G be a locally compact group, and let A and B be G-algebras; the Baaj-Skandalis duality comprises an isomorphism between $KK^G(A, B)$ and $KK^{C_r^*(G)}(A \rtimes_r G, B \rtimes_r G)$, along with a similar result for $\hat{G}$-algebras, i.e., an isomorphism between $KK^{C_r^*(G)}(A, B)$ and $KK^G(A \rtimes_r \hat{G}, B \rtimes_r \hat{G})$.

We now prove this result in E-theory. First we need a lemma.

Lemma 3.21 *For every exact group G, and every equivariant short exact sequence of $\hat{G}$-algebras $0 \to J \xrightarrow{i} B \xrightarrow{p} A \to 0$, the sequence $0 \to J \rtimes_r \hat{G} \xrightarrow{i \times 1} B \rtimes_r \hat{G} \xrightarrow{p \times 1} A \rtimes_r \hat{G} \to 0$ of reduced cross product by the coactions of G, is also exact.*

Proof. We have to prove the exactness in the middle, or more precisely the inclusion $\mathrm{im}(i \times 1) \supseteq \ker(i \times 1)$. Suppose that $\mathrm{im}(i \times 1)$ is a proper ideal of $\ker(i \times 1)$, and take the cross product by the action of G. If follows that $\mathrm{im}(i \times 1) \rtimes_r G$ is a proper ideal of $\ker(p \times 1) \rtimes_r G$. Identify $\mathrm{im}(i \times 1) \rtimes_r G$ with $\mathrm{im}\, i \otimes \mathcal{K}(L^2(G))$, and $\ker(p \times 1) \rtimes_r G$ with $\ker p \otimes \mathcal{K}(L^2(G))$, and a contradiction follows. $\square$

Theorem 3.22 *Let G be an exact group, and let A and B be G-algebras; there is an isomorphism*

$$E^G(A, B) \simeq E^{C_r^*(G)}(A \rtimes_r G, B \rtimes_r G).$$

If C and D are $\hat{G}$-algebras, there is an isomorphism

$$E^{C_r^*(G)}(C, D) \simeq E^G(C \rtimes_r \hat{G}, D \rtimes_r \hat{G}).$$

Proof. We shall define a map

$$J_G : \{A, B\}^{\overline{C_0(G)}} \to \{A \rtimes_r G, B \rtimes_r G\}^{\overline{C_r^*(G)}}$$

and a map

$$J_{\hat{G}} : \{C, D\}^{\overline{C_r^*(G)}} \to \{A \rtimes_r \hat{G}, B \rtimes_r \hat{G}\}^{\overline{C_0(G)}},$$

such that $J_{\hat{G}} \circ J_G(\varphi) = \varphi \otimes \mathrm{id}_{\mathcal{K}(L^2(G))}$ for every asymptotic morphism $\varphi \in \{A, B\}^{\overline{C_0(G)}}$ and such that $J_G \circ J_{\hat{G}}(\psi) = \psi \otimes \mathrm{id}_{\mathcal{K}(L^2(G))}$ for every asymptotic morphism $\psi \in \{C, D\}^{\overline{C_r^*(G)}}$.

First, observe that if α and β are two exterior equivalent actions of G on an algebra A, then the corresponding cross product $A \rtimes_r G$ and $A \rtimes_r G$ are isomorphic. A similar property holds for coactions of groups. This can be proved either by a direct calculation, or be seen as a particular case of [2, Proposition 7.6].

If follows that cross product is a functor $\cdot \rtimes_r G : \overline{C_0(G)}-\mathrm{alg} \to \overline{C_r^*(G)}-\mathrm{alg}$. Also, the $\hat{G}$-algebras give rise to a subcategory of $\overline{C_r^*(G)}-\mathrm{alg}$, a category that we denote $\overline{\hat{G}-\mathrm{alg}}$, and the cross product by coactions of G, gives a functor $\cdot \rtimes_r \hat{G} : \overline{\hat{G}-\mathrm{alg}} \to \overline{C_0(G)}-\mathrm{alg}$.

The question of which sufficient condition a functor F has to fulfill in order to have a natural extension to the category of homotopy classes of asymptotic morphisms, has been studied in [13, Chapter 3]. Exactness plays an important role, and the argument we used in Lemma 2.5 is quite similar to the general setting quoted above.

Thanks to the exactness of G, and the lemma above, if follows the existence of J_G and $J_{\hat{G}}$ extending the functors cross product.

The condition $J_{\hat{G}} \circ J_G(\varphi) = \varphi \otimes \mathrm{id}_{\mathcal{K}(L^2(G))}$ for every $\varphi \in \{A, B\}^{\overline{C_0(G)}}$, holds if φ is a $*$-homomorphism. Moreover, we can regard this equality as a natural transformation of two functors from the category $\overline{C_0(G)}-\mathrm{alg}$ to itself. Extension of functors to asymptotic morphisms comes along with extension of natural transformation between them, as stated by [13, Proposition 3.6]. Hence this equality holds for every $\varphi \in \{A, B\}^{\overline{C_0(G)}}$. A similar argument applies for the composition $J_G \circ J_{\hat{G}}$. The maps J_G and $J_{\hat{G}}$ induce maps in E-theory $E^G(A, B) \to E^{C_r^*(G)}(A \rtimes_r G, B \rtimes_r G)$ and $E^{C_r^*(G)}(C, D) \simeq E^G(C \rtimes_r \hat{G}, D \rtimes_r \hat{G})$ which are inverse to each other. $\square$

Remark 3.23 As explained in [2, Remark 7.7(b)], the duality isomorphism holds in a more general setting, that of a pair of dual Hopf C^*-algebras associated to a multiplicative unitary. The proofs of the theorem and of the lemma preceding it adapts to this case, provided that we assume appropriate exactness condition.

4 The universal property of KK^S

In this section we prove the existence of a natural transformation from equivariant KK-theory to equivariant E-theory. To this end, we prove an universal property

which characterizes KK-theory as a category which is stable, homotopy invariant and split exact. A similar result was proved in the non equivariant case by Higson in [14] based on previous work by Cuntz, and in the group case by Thomsen in [24]. The proof is based on a description of the KK-theory in terms of quasi-morphisms, commonly referred as the Cuntz' picture.

Let us first remind the notion of a Hilbert module endowed with a coaction of a Hopf C^*-algebra. Let S be a Hopf C^*-algebra, and assume that A is an S-algebra with coaction $\delta_A : A \to \tilde{M}(A \otimes S)$ and let E be a Hilbert A-module. Identify the Hilbert $A \otimes S$-module $E \otimes S$ with $\mathcal{K}(A \otimes S, E \otimes S)$ and denote by

$$\tilde{M}(E \otimes S) = \{T \in \mathcal{L}(A \otimes S, E \otimes S)|\ \forall s \in S\ (1_E \otimes s)T \text{ and } T(1_B \otimes s) \in E \otimes S\},$$

the S-multipliers of the module E (see section 1.4 from [11] for more details on this construction). It is a Hilbert $\tilde{M}(A \otimes S)$-module with scalar product given by $< T_1, T_2 >= T_1^* T_2 \in \tilde{M}(A \otimes S) \subset \mathcal{L}(A \otimes S)$. A coaction of S on E_A is given by a linear map $\delta_E : E \to \tilde{M}(E \otimes S)$ such that

1. $\delta_E(e)\delta_A(a) = \delta_E(ea)$,

2. $\delta_A(< e, f >) =< \delta_E(e), \delta_E(f) >$ for all $a \in A, e, f \in E$,

3. $\delta_E(E)(A \otimes S)$ is dense in $E \otimes S$ and

4. $(\mathrm{id}_E \otimes \delta_A) \circ \delta_A = (\delta_E \otimes \mathrm{id}_S) \circ \delta_A$.

The last condition is about the extensions of these maps to $\mathcal{L}(A \otimes S, E \otimes S) \to \mathcal{L}(A \otimes S \otimes S, E \otimes S \otimes S)$. A second way of defining a coaction is like follows: denote by $T_e \in \mathcal{L}(A \otimes S, E \otimes_{\delta_A} (A \otimes S))$ the operator given by $T_e(x) = e \otimes_{\delta_A} x$ for $x \in A \otimes S$, and for every $e \in E$. An unitary $V \in \mathcal{L}(E \otimes_{\delta_A} (A \otimes S), E \otimes S)$ is *admissible* if for every $e \in E$, $VT_e \in \tilde{M}(E \otimes S)$ and $(V \otimes \mathrm{id}_S)(V \otimes_{\delta_A \otimes \mathrm{id}_S} 1) = V \otimes_{\mathrm{id}_A \otimes \delta_S} 1 \in \mathcal{L}(E \otimes_{\delta_A^2} (A \otimes S \otimes S), E \otimes S \otimes S)$. There is a one-to-one correspondence between the coactions on E and the admissible unitaries [1, Proposition 2.4].

Assume that A and B are S-algebras, and that E is a Hilbert S-B-module; an $*$-homomorphism $\pi : A \to \mathcal{L}(E)$ is S-equivariant if for all $a \in A$ and $e \in E$, $\delta_E(\pi(a)e) = (\pi \otimes \mathrm{id}_S)(\delta_A)(a) \circ \delta_E(e)$.

Let us first recall the definition of the equivariant KK-theory of Baaj and Skandalis. Let A, B be two graded S-algebras. A Kasparov triple (E, π, T) is given by a Hilbert S-B-module E, a grading preserving S-equivariant representation $\pi : A \to \mathcal{L}(E)$, and a degree one operator $T \in \mathcal{L}(E)$ such that

(i) $\pi(a)(T - T^*) \in \mathcal{K}(E)$;

(ii) $\pi(a)(T^2 - 1) \in \mathcal{K}(E)$;

(iii) $[T, \pi(a)] \in \mathcal{K}(E)$;

(iv) for all $x \in A \otimes S$, $(\pi \otimes \mathrm{id}_S)(x)(F \otimes_{\mathbb{C}} 1 - V(F\hat{\otimes}_{\delta_B}1)V^*) \in \mathcal{K}(E \otimes S)$, where $V \in \mathcal{L}(E \otimes_{\delta_B} (B \otimes S), E \otimes S)$ is the unitary defining the coaction of S on the Hilbert module E.

The last condition can also be written as follows $(F \otimes_{\mathbb{C}} 1 - V(F\hat{\otimes}_{\delta_B}1)V^*)(\pi \otimes \mathrm{id}_S)(x) \in \mathcal{K}(E \otimes S)$, for all $x \in A \otimes S$.

Such a triple is called degenerate if $\pi(a)(T - T^*) = \pi(a)(T^2 - 1) = [T, \pi(a)] = 0$ for all $a \in A$ and if $(\pi \otimes \mathrm{id}_S)(x)(F \otimes_{\mathbb{C}} 1 - V(F\hat{\otimes}_{\delta_B}1)V^*) = 0$, for all $x \in A \otimes S$. The set of unitary equivalence classes of Kasparov triples form a semigroup. The group $KK^S(A, B)$ is defined as the quotient of this semigroup by a homotopy relation which is defined using triples for the pair of algebras $(A, B[0, 1])$. Degenerate triples are homotopic to zero.

From now on, algebras are trivially graded.

Definition 4.1 An S-quasi-morphism is a pair $(\varphi_+, \varphi_-) : A \to M(B \otimes \mathcal{K})$ of quasi-unital morphisms, along with a pair $(V_+, V_-) \in M(B \otimes \mathcal{K} \otimes S)$ of $\delta_B \otimes \mathrm{id}_{\mathcal{K}}$-cocycles such that

(i) $\varphi_+(a) - \varphi_-(a) \in B \otimes \mathcal{K}$, for all $a \in A$;

(ii) $\varphi_\pm$ are δ_A-$\delta_{B\otimes\mathcal{K}}^\pm$-equivariant where $\delta_{B\otimes\mathcal{K}}^\pm$ denote the coaction $\delta_{B\otimes\mathcal{K}}^\pm(\cdot) := V_\pm(\delta_B(\cdot) \otimes \mathrm{id}_{\mathcal{K}})V_\pm^*$;

(iii) $V_+ - V_- \in \tilde{M}(B \otimes \mathcal{K} \otimes S)$.

An S-quasi-morphism is called *degenerate* if $\varphi_+ = \varphi_- = 0$.

Two S-quasi-morphisms $(\varphi_\pm, U_\pm)$ and $(\psi_\pm, V_\pm)$ are *isomorphic* if there is a $\delta_B \otimes \mathrm{id}_{\mathcal{K}}$-equivariant automorphism Θ of $B \otimes \mathcal{K}$ such that $\psi_\pm = \Theta \circ \varphi_\pm$ and such that $V_\pm = (\Theta \otimes \mathrm{id}_S) \circ U_\pm$. Two S-quasi-morphisms $(\varphi_\pm^0, U_\pm^0)$ and $(\varphi_\pm^1, U_\pm^1)$ are *homotopic* if there is an S-quasi-morphism $(\Phi_\pm, U_\pm) : A \to M(B \otimes \mathcal{K} \otimes C[0, 1])$ such that

$$(\varphi_\pm^i, U_\pm^i) = (ev_i \circ \Phi_\pm, (ev_i \otimes \mathrm{id}_S) \circ U_\pm),$$

where $ev_i : M(B \otimes \mathcal{K} \otimes C[0, 1]) \to M(B \otimes \mathcal{K})$ denote the evaluations in $i = 0$ and in $i = 1$. The sum the S-quasi-morphisms $(\varphi_\pm, U_\pm)$ and $(\psi_\pm, V_\pm)$ is defined using an isomorphism $M_2(\mathcal{K}) \simeq \mathcal{K}$ as $(\varphi_\pm \oplus \psi_\pm, U_\pm \oplus V_\pm)$; its homotopy class does not depend on the choice made. Finally, two S-quasi-morphisms $(\varphi_\pm, U_\pm)$ and $(\psi_\pm, V_\pm)$ are *equivalent* if there are degenerate S-quasi-morphisms $(\varphi_\pm', U_\pm')$ and $(\psi_\pm', V_\pm')$ such that $(\varphi_\pm, U_\pm) \oplus (\varphi_\pm', U_\pm')$ and $(\psi_\pm, V_\pm) \oplus (\psi_\pm', V_\pm')$ are homotopic.

Theorem 4.2 $KK^S(A, B)$ *is isomorphic with the group of equivalence classes of S-quasi-morphisms from A to B.*

Proof. We assume that B is trivially stable, this in order to eliminate from what follows an extra copy of $\mathcal{K}$, with the trivial coaction on it, this when we think that this will not hinder the argument. Let $(\varphi_\pm, V_\pm)$ be a S-quasi-morphism, we associate to it the following triple (E, π, T):

- the Hilbert module E is the direct sum $B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}$ with the coaction of S given by $V_+(\delta_B \otimes \mathrm{id}_\mathcal{K}) \oplus V_-(\delta_B \otimes \mathrm{id}_\mathcal{K})$.

- the representation $\pi : A \to \mathcal{L}_{B \otimes \mathcal{K}}(E)$ and the operator $F \in \mathcal{L}_{B \otimes \mathcal{K}}(E)$ respectively defined by

$$\pi(a) = \begin{pmatrix} \varphi_+(a) & 0 \\ 0 & \varphi_-(a) \end{pmatrix} \text{ and by } F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that the condition $V_+ - V_- \in \tilde{M}(B \otimes \mathcal{K} \otimes S)$ writes as the equivariance condition for this particular triple. In this way one associates a degenerate triple to a degenerate S-quasi-morphism; it agrees with the direct sums and with the homotopies of S-quasi-morphisms and of equivariant Kasparov's triples. Hence it defines a map from the set of equivalence classes of S-quasi-morphisms $(\varphi_\pm, V_\pm) : A \to M(B \otimes \mathcal{K})$ to the group $KK^S(A, B)$.

Conversely, take $(E, \pi, T) \in KK^S(A, B \otimes \mathcal{K})$. One can assume that the representation π is nondegenerate, as the argument from [3, Proposition 18.3.6] applies to the equivariant case too. The idea of the proof is first to replace the Hilbert module E with the Hilbert module $B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}$ and then to replace the operator T with the operator $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In then nonequivariant case, this is described in detail [3, Section 17.6] and in the group case it is done by Thomsen in [24]. We describe how this transformations work in our case.

Denote by $V : E \to \tilde{M}(E \otimes S)$ the unitary implementing the coaction of S on the Hilbert module E. The coaction on the $B \otimes \mathcal{K}$-module $B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}$ is the standard one.

The triple $(B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}, 0, 0) \in KK^S(A, B \otimes \mathcal{K})$ is degenerate thus

$$(E, \pi, T) \oplus (B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}, 0, 0) = (E, \pi, T).$$

The Kasparov stabilization theorem states that there is a non equivariant graded isomorphism of Hilbert $B \otimes \mathcal{K}$-modules

$$E \oplus l^2(B \otimes \mathcal{K}) \oplus l^2(B \otimes \mathcal{K}) \simeq l^2(B \otimes \mathcal{K}) \oplus l^2(B \otimes \mathcal{K}).$$

Moreover, because $B \otimes \mathcal{K}$ is stable, there is a graded isomorphism of Hilbert $B \otimes \mathcal{K}$-modules between $l^2(B \otimes \mathcal{K}) \oplus l^2(B \otimes \mathcal{K})$ and $B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}$ as shown in [16, Lemma 1.3.2]; denote by

$$\Psi : E \oplus B \otimes \mathcal{K} \oplus B \otimes \mathcal{K} \to B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}$$

the resulting isomorphism. Let W be the $*$-homomorphism making the following diagram commutative:

$$\begin{array}{ccc}
E \oplus (B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}) & \xrightarrow{\ \Psi\ } & B \otimes \mathcal{K} \oplus B \otimes \mathcal{K} \\
\Big\downarrow{\scriptstyle V \oplus \delta_B \otimes \mathrm{id}_\mathcal{K} \oplus \delta_B \otimes \mathrm{id}_\mathcal{K}} & & \Big\downarrow{\scriptstyle W} \\
\tilde{M}(E \otimes S \oplus (B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}) \otimes S) & \xrightarrow{\Psi \otimes \mathrm{id}_S} & \tilde{M}((B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}) \otimes S);
\end{array}$$

it is a coaction on $B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}$, in general different from the standard one, as we cannot assume that the stabilization morphism Ψ is S-equivariant.

This takes care of the Hilbert module. We can further simplify the resulting triple and obtain a triple

$$(B \otimes \mathcal{K} \oplus B \otimes \mathcal{K}, \varphi, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

Note that the representation φ is quasi-unital.

The action on this module has the form $V = (V_+, V_-)$ with V_+, V_- denoting the coactions of S on the Hilbert $B \otimes \mathcal{K}$-module $B \otimes \mathcal{K}$. Denote $V_+, V_- \in \mathcal{L}(B \otimes \mathcal{K} \otimes S, B \otimes \mathcal{K} \otimes S)$ the unitaries defining this action, and set $U_\pm \in U(M(B \otimes \mathcal{K} \otimes S))$, the $\delta_B \otimes \mathrm{id}_\mathcal{K}$-cocycles

$$U_\pm = V_\pm \circ (\delta_B \otimes \mathrm{id}_\mathcal{K})^*.$$

Take $\varphi = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{pmatrix}$; $\varphi_\pm$ is δ_A-$U_\pm(\delta_B \otimes \mathrm{id}_\mathcal{K})U_\pm^*$-equivariant. It follows from the condition $[\varphi(a), F] \in B \otimes \mathcal{K}$ that $\varphi_+(a) - \varphi_-(a) \in B \otimes \mathcal{K}$, and it follows from the equivariance condition that $U_+ - U_- \in \tilde{M}(B \otimes \mathcal{K} \otimes S)$, thus $(\varphi_\pm, U_\pm)$ is an S-quasi-morphism.

The rest of the proof, follows like in the group case. $\square$

That $KK^S(A, \cdot)$ is itself a functor which is stable, homotopy invariant, and split-exact. Stability is best express through Morita equivalence: an equivariant imprimitivity bimodule provides an invertible element in equivariant KK-theory. Homotopy invariance follows from the definitions. The argument from [24] adapts without changes to prove split-exactness.

Proposition 4.3 (*Universal property of KK^S*) *Let $F : S-\mathrm{alg} \to Ab$ be a covariant functor which is stable, homotopy invariant, and split-exact. Then for every S-algebra A and every element $d \in F(A)$ there exists an unique natural transformation $T_A : KK^S(A, \cdot) \to F(\cdot)$ such that $T_A(1_A) = d$.*

Proof. We indicate how an element $x \in KK^S(A, B)$ defines an application $L(x) :$ $F(A) \to F(B)$, as we do it a bit differently from [24]. The rest of the proof, follows like in the non-equivariant case.

Let $(\varphi_\pm, V_\pm)$ be a S-quasi-morphism representing x.

Consider the coaction $\bar{\delta}_B$ on $M_2(B \otimes \mathcal{K})$ which on the upper left corner restricts to δ_B^+ and on the lower right corner restricts to δ_B^-. This action is exterior equivalent to δ_B, with cocycle $\mathrm{diag}(V_+, V_-)$, hence there is an isomorphism

$$I_B : F(B, \delta_B) \to F(M_2(B \otimes \mathcal{K}), \bar{\delta}_B).$$

The maps $i_B^+(b) = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$ and $i_B^-(b) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, induce respectively the isomorphisms $i_{B*}^+ : F(B \otimes \mathcal{K}, \delta_B^+) \to F(M_2(B \otimes \mathcal{K}), \bar{\delta}_B)$ and $i_{B*}^- : F(B \otimes \mathcal{K}, \delta_B^-) \to F(M_2(B \otimes \mathcal{K}), \bar{\delta}_B)$.

Denote by A_x the subalgebra of $A \oplus M_2(M(B \otimes \mathcal{K}))$ of elements (a, m) such that: $\varphi_+(a) - m_{11} \in B \otimes \mathcal{K}$, $\varphi_-(a) - m_{22} \in B \otimes \mathcal{K}$ and $m_{12}, m_{21} \in B \otimes \mathcal{K}$.

It is an S-algebra with coaction $(\delta_A, \bar{\delta}_B)$, and it fits into an equivariant short exact sequence

$$0 \longrightarrow (M_2(B \otimes \mathcal{K}), \bar{\delta}_B) \overset{j}{\longrightarrow} A_x \overset{p}{\longrightarrow} (A, \delta_A) \longrightarrow 0$$

with maps given by $j(b) = (0, b)$ and by $p(a, m) = a$.

This short exact sequence splits equivariantly into two different ways by the $*$-homomorphisms $s_\pm : A \to A_x$, given by

$$s_+(a) = \left(a, \begin{pmatrix} \varphi_+(a) & 0 \\ 0 & 0 \end{pmatrix} \right) \text{ and } s_-(a) = \left(a, \begin{pmatrix} 0 & 0 \\ 0 & \varphi_-(a) \end{pmatrix} \right).$$

Define $L(x) : F(A) \to F(B)$ as the composition

$$\Phi : F(A) \to F(A_x) \to F(M_2(B \otimes \mathcal{K}), \bar{\delta}_B) \to F(B, \delta_B)$$

given by $\Phi = I_{B*}^{-1} \circ j_*^{-1} \circ (s_{+*} - s_{-*})$. $\square$

Remark 4.4 Let us now describe the product in KK^S using Cuntz' picture, a question implicit in [24]. Let A, B, C be S-algebras and consider $(\varphi_\pm, U_\pm) \in KK^S(A, B)$ and $(\psi_\pm, V_\pm) \in KK^S(A, B)$, the product of the two elements is given by the S-quasimorphism

$$(\phi_+ \circ \varphi_+ \oplus \phi_- \circ \varphi_-, \phi_- \circ \varphi_+ \oplus \phi_+ \circ \varphi_-)$$

with the associated cocycle

$$(V_+ \varphi_+^\sharp(U_+) \oplus V_- \varphi_-^\sharp(U_-), V_- \varphi_+^\sharp(U_+) \oplus V_+ \varphi_-^\sharp(U_-)).$$

This defines a product in the category KK^S, and the claim follows from the fact that such a product is unique. This in itself is a consequence of the universal property.

Corollary 4.5 *There is a natural transformation from the category KK^S to the category E^S.*

5 Appendix: equivariant stabilization

In this appendix we examine various ways in which stability of a functor can be expressed.

Let $F : S-\text{alg} \to Ab$ be a functor from the category of S-algebras to abelian groups. The following definition is the notion of stability used by Thomsen to characterize KK-theory.

Definition 5.1 The functor $F : S-\text{alg} \to Ab$ is stable if for every S-algebra (A, δ_A) and every compatible action $\delta_{A \otimes \mathcal{K}}$ on $A \otimes \mathcal{K}$, the compatibility morphism $a \to a \otimes e$ induces an isomorphism $F(A, \delta_A) \simeq F(A \otimes \mathcal{K}, \delta_{A \otimes \mathcal{K}})$.

Remark that if F is stable and if V is a δ_A-cocycle then $F(A, \delta_A) \simeq F(A, \delta_A^V)$, as by the lemma 3.4 they are both compatible with $(M_2(A) \otimes \mathcal{K}, \delta_{M_2(A)} \otimes \text{id}_{\mathcal{K}})$.

In the non equivariant case, stability can be stated in terms of Morita equivalence. Let us recall this notion: two algebras A and B are Morita equivalent if there is a Hilbert B-module which is full, i.e. such that $< E, E >= B$, and such that $A \simeq \mathcal{K}_B(E)$; the module E is called imprimitivity bimodule.

The Brown-Green-Rieffel theorem ([22]) states that two C^*-algebras A and B are Morita equivalent if and only if they are stably isomorphic, that is, if $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$. Hence, in the non-equivariant case, a functor is stable if and only if it is Morita invariant in the obvious sense.

The notion of Morita equivalence extends naturally to the equivariant case: for a Hopf-C^*-algebra S and two S-algebras A and B assume that there is a Hilbert S-B-module E which is full and such that the isomorphism $A \simeq \mathcal{K}_B(E)$ is S-equivariant.

Examples.1. Let G be a group and (A, α) be a G-algebra; then (A, α) and $(A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \lambda_G)$ are G-Morita equivalent with imprimitivity bimodule $L^2(G, A)$.
2. Let (A, δ_A) be an S-algebra; it is Morita equivalent to (A, δ_A') if and only if the actions δ_A and δ_A' are exterior equivalent. To see this, note that the linking algebra is in this case $M_2(A)$ and that the action on diagonal elements is the diagonal $\begin{pmatrix} \delta_A & 0 \\ 0 & \delta_A' \end{pmatrix}$, hence the claim follows from the lemma 3.4.
The imprimitivity bimodule is the algebra A itself, seen as a Hilbert A-module, endowed with the coaction $\delta : A \to \tilde{M}(A \otimes S)$, $\delta(a) = V\delta_A(a)$, where V denotes the δ_A-cocycle for which $\delta_A'(\cdot) = V\delta_A(\cdot)V^*$.
3. Let (A, δ_A) be an S-algebra; a coaction $(A \otimes \mathcal{K}, \delta_{A \otimes \mathcal{K}})$ is compatible with δ_A if and only if the two algebras are Morita equivalent through an imprimitivity bimodule $A \oplus A_A^\infty$ with the coaction on the first factor A given by δ_A. The nontrivial implication follows from [1, Proposition 2.7 (a)].

Even though not needed here, it is worth mentioning that Morita equivalence appears as more natural from the point of view of the categories involved (see [11]).

The following is an equivariant version of the Brown-Green-Rieffel theorem, which shows that the notion of stability we use, is also equivalent to equivariant Morita equivalence. In the group case, it is due to Combes [5]; his proof is different though.

Theorem 5.2 *Let (A, δ_A) and (B, δ_B) be S-algebras, S-equivarianly Morita equivalent. There is a coaction $\delta_{B \otimes \mathcal{K}}$ of S on $B \otimes \mathcal{K}$, which is compatible with δ_B, such that the algebras $(A \otimes \mathcal{K}, \delta_A \otimes \mathrm{id}_{\mathcal{K}})$ and $(B \otimes \mathcal{K}, \delta_{B \otimes \mathcal{K}})$ are S-isomorphic.*

Proof. Let us first remaind that the proof of the (non-equivariant) Brown-Green-Rieffel theorem is based on the following result: for every full Hilbert B-module, there is an isomorphism of Hilbert B-modules between E^∞ and B^∞. If follows that if A and B are Morita equivalent with imprimitivity bimodule E, then

$$A \otimes \mathcal{K} \simeq \mathcal{K}_B(E^\infty) \simeq \mathcal{K}_B(B^\infty) \simeq B \otimes \mathcal{K}.$$

Assume that A, B and E are endowed with coactions of S, denoted respectively by δ_A, δ_B, and δ_E. The first isomorphism above is equivariant for the coaction $\delta_A \otimes \mathrm{id}_{\mathcal{K}}$ on $A \otimes \mathcal{K}$ and the coaction on $\mathcal{K}_B(E^\infty) \simeq \mathcal{K}_B(E) \otimes \mathcal{K}$ induced by the coaction δ_E on E and the trivial coaction on $\mathcal{K}$.

Consider the map β making the following diagram commutative:

$$
\begin{array}{ccc}
E^\infty & \xrightarrow{\ \ T\ \ } & B^\infty \\
{\scriptstyle \delta_E^\infty} \downarrow & & \downarrow {\scriptstyle \beta} \\
\tilde{M}(E^\infty \otimes S) & \xrightarrow{\ T \otimes \mathrm{id}_S\ } & \tilde{M}(B^\infty \otimes S)
\end{array}
$$

where $T : E^\infty \to B^\infty$ denotes a (non-equivariant) isomorphism of Hilbert B-modules. It is easy to check that β defines a coaction on the Hilbert S-B-module B^∞, and we claim that it induces on the algebra $\mathcal{K}_B(B^\infty) \simeq B \otimes \mathcal{K}$ a coaction $\delta_{B \otimes \mathcal{K}}$ which is compatible with δ_B, and for which the above isomorphism $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$ is equivariant. Consider the isomorphism of Hilbert B-modules

$$T \oplus \mathrm{id}_{B^\infty} : E^\infty \oplus B^\infty \to B^\infty \oplus B^\infty;$$

it is $\delta_E^\infty \oplus \delta_B^\infty$-$\beta \oplus \delta_B^\infty$-equivariant. This isomorphism induces an isomorphism of S-algebras between the corresponding algebras of compact operators, and hence there is an action on $\mathcal{K}(E^\infty \oplus B^\infty) \simeq M_2(B \otimes \mathcal{K})$ which on left-upper corner is $\delta_{B \otimes \mathcal{K}}$ and on the lower-right corner is $\delta_B \otimes \mathrm{id}_{\mathcal{K}}$, which proves compatibility. The rest is obvious. $\square$

To summarize the discussion above, stabilization can be define in three equivalent ways:

1. (stable) $F(A, \delta_A) \simeq F(A \otimes \mathcal{K}, \delta_{A \otimes \mathcal{K}})$ for every S-algebra (A, δ_A) and every compatible coaction $\delta_{A \otimes \mathcal{K}}$ on $A \otimes \mathcal{K}$,

2. (Morita equivalence) $F(A, \delta_A) \simeq F(B, \delta_B)$ for every S-Morita equivalent pair of S-algebras (A, δ_A) and (B, δ_B), and

3. (exterior equivalence along with trivial stability) $F(A, \delta_A) \simeq F(A, \delta'_A)$ for a pair of exterior equivalent coactions, and $F(A, \delta_A) \simeq F(A \otimes \mathcal{K}, \delta_A \otimes \mathrm{id}_K)$.

Let us now take a closer look at the case of a group G action.

Lemma 5.3 *A functor F defined on the category of G-algebras is stable if and only if $F(\cdot) \simeq F(\cdot \otimes \mathcal{K}_G)$.*

Proof. Let A be a G-algebra; A and $A \otimes \mathcal{K}_G$ are G-Morita equivalent so the condition is necessary for stability.

Assume now that $F(A, \alpha) \simeq F(A \otimes \mathcal{K}_G, \alpha \otimes \lambda_G)$ for every action α of G on A. Note first that $(\mathcal{K}_G, \lambda_G) \simeq (\mathcal{K}_G \otimes \mathcal{K}, \lambda_G \otimes \mathrm{id}_{\mathcal{K}})$, hence we only need to check that if α and α' are two exterior actions of G on A, then $F(A, \alpha) \simeq F(A, \alpha')$. Write $\alpha'_g(\cdot) = u_g \alpha_g(\cdot) u_g^*$; and consider the (A, α')-(A, α)-imprimitivity bimodule A, with its coaction γ given by $\gamma_g(a) = u_g \alpha_g(a)$, for $a \in A$ and $g \in G$. The Hilbert G-A-modules (A_A, γ) and (A_A, α) are non equivariantly isomorphic, thus $(L^2(G, A)_A, \lambda_G \otimes \gamma)$ and $(L^2(G, A)_A, \lambda_G \otimes \alpha)$ are G-isomorphic. Take their compact operators, which are respectively $(A \otimes \mathcal{K}_G, \alpha' \otimes \lambda_G)$ and $(A \otimes \mathcal{K}_G, \alpha \otimes \lambda_G)$, are G-isomorphic, hence $F(A, \alpha) \simeq F(A, \alpha')$.□

If follows that given a functor F, one can replace it with the functor $\bar{F}(\cdot) := F(\cdot \otimes \mathcal{K}_G)$, which is stable.

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